Localizing Nonlinear Eigenvalues: Theory and Applications

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Vibrations are everywhere, and so too are the eigenvalues associated with them. As mathematical models invade more and more disciplines, we can anticipate a demand for eigenvalue calculations in an ever richer variety of contexts.

– Beresford Parlett, The Symmetric Eigenvalue Problem
Why Nonlinear Eigenvalues?

\[
y' - Ay = 0 \quad \Rightarrow \quad (\lambda I - A)v = 0
\]
\[
y'' + By' + Ky = 0 \quad \Rightarrow \quad (\lambda^2 I + \lambda B + K)v = 0
\]
\[
y' - Ay - By(t - 1) = 0 \quad \Rightarrow \quad (\lambda I - A - Be^{-\lambda})v = 0
\]

- Higher-order ODEs
- Dynamic element formulations
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
My motivation

\[
T(\omega)v \equiv (K - \omega^2 M + G(\omega)) v = 0
\]

**Wanted:** Perturbation theory justifying a terrible estimate of \(G(\omega)\)
Nonlinear eigenvalue problem

\[ T(\lambda)v = 0, \quad v \neq 0. \]

where

- \( T : \Omega \to \mathbb{C}^{n \times n} \) analytic, \( \Omega \subset \mathbb{C} \) simply connected
- Regularity: \( \det(T) \neq 0 \)

Nonlinear spectrum: \( \Lambda(T) = \{ z \in \Omega : T(z) \text{ singular} \} \).

**Goal:** Use analyticity to *compare* and to *count*
Winding, Rouché, and Gohberg-Sigal

Analytic: $f, g : \Omega \to \mathbb{C}$

Winding #: $\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz$

Theorem: Rouché:

- $|g| < |f|$ on $\Gamma \implies$ same # zeros of $f, f + g$

- $T, E : \Omega \to \mathbb{C}^{n \times n}$
- $\text{tr} \left( \frac{1}{2\pi i} \int_{\Gamma} T(z)^{-1} T'(z) \, dz \right)$

Gohberg-Sigal 1971:

- $\|T^{-1}E\| < 1$ on $\Gamma \implies$ same # eigs of $T, T + E$
Comparing NEPs

Suppose

\[ T, E : \Omega \rightarrow \mathbb{C}^{n \times n} \text{ analytic} \]
\[ \Gamma \subset \Omega \text{ a simple closed contour} \]
\[ T(z) + sE(z) \text{ nonsingular} \quad \forall s \in [0, 1], z \in \Gamma \]

Then \( T \) and \( T + E \) have the same number of eigenvalues inside \( \Gamma \).

**Pf:** Constant winding number around \( \Gamma \).
Nonlinear pseudospectra

\[ \Lambda_\epsilon(T) \equiv \{ z \in \Omega : \|T(z)^{-1}\| > \epsilon^{-1} \} \]
$E$ analytic, $\| E(z) \| < \epsilon$ on $\Omega_\epsilon$. Then

$$\Lambda(T + E) \cap \Omega_\epsilon \subset \Lambda_\epsilon(T) \cap \Omega_\epsilon$$

Also, if $\mathcal{U}_\epsilon$ a component of $\Lambda_\epsilon$ and $\bar{\mathcal{U}}_\epsilon \subset \Omega_\epsilon$, then

$$| \Lambda(T + E) \cap \mathcal{U}_\epsilon | = | \Lambda(T) \cap \mathcal{U}_\epsilon |$$
Pseudospectral comparison

- Most useful when $T$ is linear
- Even then, can be expensive to compute!
- What about related tools?

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The Gershgorin picture (linear case)

\[ A = D + F, \quad D = \text{diag}(d_i), \quad \rho_i = \sum_j |f_{ij}| \]
Gershgorin $(+\epsilon)$

Write $A = D + F$, $D = \text{diag}(d_1, \ldots, d_n)$. Gershgorin disks are:

$$G_i = \left\{ z \in \mathbb{C} : |z - d_i| \leq \sum_j |f_{ij}| \right\}.$$

Useful facts:

- Spectrum of $A$ lies in $\bigcup_{i=1}^{m} G_i$
- $\bigcup_{i \in \mathcal{I}} G_i$ disjoint from other disks $\implies$ contains $|\mathcal{I}|$ eigenvalues.

Pf:

$A - zI$ strictly diagonally dominant outside $\bigcup_{i=1}^{m} G_i$.

Eigenvalues of $D - sF$, $0 \leq s \leq 1$, are continuous.
Nonlinear Gershgorin

Write \( T(z) = D(z) + F(z) \). Gershgorin regions are

\[
G_i = \left\{ z \in \mathbb{C} : |d_i(z)| \leq \sum_j |f_{ij}(z)| \right\}.
\]

Useful facts:

- Spectrum of \( T \) lies in \( \bigcup_{i=1}^m G_i \)
- Bdd connected component of \( \bigcup_{i=1}^m G_i \) strictly in \( \Omega \) \( \implies \) same number of eigs of \( D \) and \( T \) in component
  \( \implies \) at least one eig per component of \( G_i \) involved

**Pf:** Strict diag dominance test + continuity of eigs
Example I: Hadeler

\[ T(\bar{z}) = (e^{\bar{z}} - 1)B + \bar{z}^2A - \alpha I, \quad A, B \in \mathbb{R}^{8 \times 8} \]
Comparison to simplified problem

Bauer-Fike idea: apply a similarity!

\[ T(z) = (e^z - 1)B + z^2 A - \alpha I \]

\[ \tilde{T}(z) = U^T T(z) U \]
\[ = (e^z - 1)D_B + z^2 I - \alpha E \]
\[ = D(z) - \alpha E \]

\[ G_i = \{ z : |\beta_i(e^z - 1) + z^2| < \rho_i \}. \]
Gershgorin regions
A different comparison

Approximate $e^z - 1$ by a Chebyshev interpolant:

$$T(z) = (e^z - 1)B + z^2A - \alpha I$$
$$\tilde{T}(z) = q(z)B + z^2A - \alpha$$

$$T(z) = \tilde{T}(z) + r(z)B$$

Linearize $\tilde{T}$ and transform both:

$$\tilde{T}(z) \mapsto D_C - zI$$
$$T(z) \mapsto D_C - zI + r(z)E$$

Restrict to $\Omega_\epsilon = \{ z : |r(z)| < \epsilon \}$:

$$G_i \subset \hat{G}_i = \{ z : |z - \mu_i| < \rho_i \epsilon \}, \quad \rho_i = \sum_j |e_{ij}|$$
Spectrum of $\tilde{T}$
\( \hat{G}_i \) for \( \epsilon < 10^{-10} \)
$\hat{G}_i$ for $\epsilon = 0.1$
$\hat{G}_i$ for $\epsilon = 1.6$
Example II: Resonance problem

\[ V_0 \]

\[ \psi(0) = 0 \]
\[ \left( -\frac{d^2}{dx^2} + V - \lambda \right) \psi = 0 \text{ on } (0, b), \]
\[ \psi'(b) = i\sqrt{\lambda}\psi(b), \]
Reduction via shooting

\[ V_0 \]

\[ \psi(0) = 0, \]

\[ R_{0a}(\lambda) \begin{bmatrix} \psi(0) \\ \psi'(0) \end{bmatrix} = \begin{bmatrix} \psi(a) \\ \psi'(a) \end{bmatrix}, \]

\[ R_{ab}(\lambda) \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix} = \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix}, \]

\[ \psi'(b) = i \sqrt{\lambda} \psi(b) \]
Reduction via shooting

First-order form:

\[
\frac{du}{dx} = \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix} u, \quad \text{where } u(x) \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}.
\]

On region \((c, d)\) where \(V\) is constant:

\[
u(d) = R_{cd}(\lambda)u(c), \quad R_{cd}(\lambda) = \exp \left( (d - c) \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix} \right)\]

Reduce resonance problem to 6D NEP:

\[
T(\lambda)u_{\text{all}} \equiv \begin{bmatrix} R_{0a}(\lambda) & -I & 0 \\ 0 & R_{ab}(\lambda) & -I \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(a) \\ u(b) \end{bmatrix} = 0.
\]
Expansion via rational approximation

\[ u(0) u(a) u(b) \times = 0 \]
Analyzing the expanded system

- \( \hat{T}(z) \) is a Schur complement in \( K - zM \)
  - So \( \Lambda(\hat{T}) \) is easy to compute.
- Or: think \( T(z) \) is a Schur complement in \( K - zM + E(z) \)
- Compare \( \hat{T}(z) \) to \( T(z) \) or compare \( K - zM + E(z) \) to \( K - zM \)
Analyzing the expanded system

Q: Can we find all eigs in a region not missing anything?

Concrete plan \((\epsilon = 10^{-8})\)

- \(T = \) shooting system
- \(\hat{T} = \) rational approximation
- Find region \(D\) with boundary \(\Gamma\) s.t.
  - \(D \subset \Omega_\epsilon\) (i.e. \(\|T - \hat{T}\| < \epsilon\) on \(D\))
  - \(\Gamma\) does not intersect \(\Lambda_\epsilon(T)\)

\(\rightarrow\) Same eigenvalue counts for \(T, \hat{T}\)
\(\rightarrow\) Eigs of \(\hat{T}\) in components of \(\Lambda_\epsilon(T)\)
  - Converse holds if \(D \subset \Omega_{\epsilon/2}\)

Can refine eigs of \(\hat{T}\) in \(D\) via Newton.
Resonance approximation
Resonance approximation
For more

Localization theorems for nonlinear eigenvalues.
David Bindel and Amanda Hood, SIMAX 34(4), 2013

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