Eigenvalue Localization and Applications

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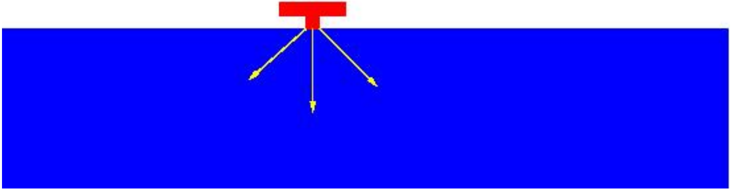
My motivation



$$T(\omega)v \equiv \left(K - \omega^2 M + G(\omega)\right) v = 0$$

Wanted: Perturbation theory justifying a terrible estimate of $G(\omega)$

Nonlinear eigenvalue problem

$$T(\lambda)v = 0, \quad v \neq 0.$$

where

- $T : \Omega \to \mathbb{C}^{n\times n}$ analytic, $\Omega \subset \mathbb{C}$ simply connected
- Regularity: $\det(T) \not\equiv 0$

Nonlinear spectrum: $\Lambda(T) = \{z \in \Omega : T(z)$ singular$\}$.

Goal: Use analyticity to *compare* and to *count*

Comparing NEPs

Suppose

$$T, E : \Omega \to \mathbb{C}^{n\times n} \text{ analytic}$$
$$\Gamma \subset \Omega \text{ a simple closed contour}$$
$$T(z) + sE(z) \text{ nonsingular } \forall s \in [0,1], z \in \Gamma$$

Then T and $T+E$ have the same number of eigenvalues inside Γ.

Pf: Constant winding number around Γ.

Nonlinear pseudospectra

Let

$$\mathcal{E} = \{E : \Omega \to \mathbb{C}^{n\times n} \text{ s.t. } E \text{ analytic}, \sup_{z\in\Omega} \|E(z)\| < \epsilon\}$$

$$\mathcal{E}_0 = \{E_0 \in \mathbb{C}^{n\times n} : \|E_0\| < \epsilon\}$$

Then define

$$\begin{aligned}\Lambda_\epsilon(T) &\equiv \{z \in \Omega : \|T(z)^{-1}\| > \epsilon^{-1}\} \\ &= \bigcup_{E\in\mathcal{E}} \Lambda(T+E) \\ &= \bigcup_{E_0\in\mathcal{E}_0} \Lambda(T+E_0).\end{aligned}$$

Many of the same features as ordinary pseudospectra.

Pseudospectral comparison

E analytic, $\|E(z)\| < \epsilon$ on Ω_ϵ. Then

$$\Lambda(T+E) \cap \Omega_\epsilon \subset \Lambda_\epsilon(T) \cap \Omega_\epsilon$$

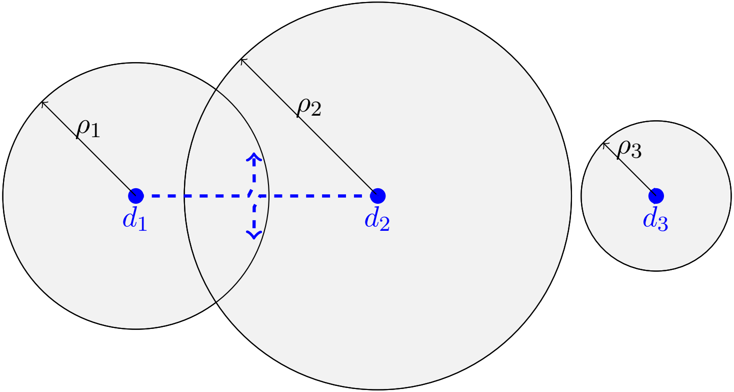
Also, if $\mathcal{U}_\epsilon$ a component of Λ_ϵ and $\bar{\mathcal{U}_\epsilon} \subset \Omega_\epsilon$, then

$$|\Lambda(T+E) \cap \mathcal{U}_\epsilon| = |\Lambda(T) \cap \mathcal{U}_\epsilon|$$

- Most useful when T is linear
- Even then, can be expensive to compute!
- What about related tools?

The Gershgorin picture (linear case)

$$A = D + F, \qquad D = \operatorname{diag}(d_i), \qquad \rho_i = \sum_j |f_{ij}|$$



Gershgorin $(+\epsilon)$

Write $A = D + F$, $D = \mathrm{diag}(d_1, \dots, d_n)$. Gershgorin disks are:

$$G_i = \left\{ z \in \mathbb{C} : |z - d_i| \leq \sum_j |f_{ij}| \right\}.$$

Useful facts:

- Spectrum of A lies in $\bigcup_{i=1}^m G_i$
- $\bigcup_{i \in \mathcal{I}} G_i$ disjoint from other disks $\implies$ contains $|\mathcal{I}|$ eigenvalues.

Pf:

$A - zI$ strictly diagonally dominant outside $\bigcup_{i=1}^m G_i$.
Eigenvalues of $D - sF$, $0 \leq s \leq 1$, are continuous.

Nonlinear Gershgorin

Write $T(z) = D(z) + F(z)$. Gershgorin *regions* are

$$G_i = \left\{ z \in \mathbb{C} : |d_i(z)| \leq \sum_j |f_{ij}(z)| \right\}.$$

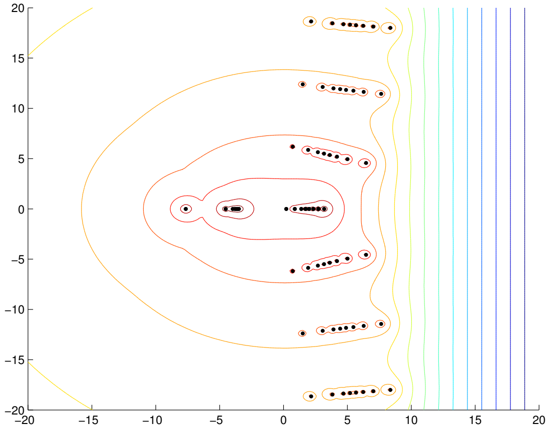
Useful facts:

- Spectrum of T lies in $\bigcup_{i=1}^m G_i$
- Bdd connected component of $\bigcup_{i=1}^m G_i$ strictly in Ω
  $\Longrightarrow$ same number of eigs of D and T in component
  $\Longrightarrow$ at least one eig per component of G_i involved

Pf: Strict diag dominance test + continuity of eigs

Example I: Hadeler

$$T(z) = (e^z - 1)B + z^2 A - \alpha I, \quad A, B \in \mathbb{R}^{8\times 8}$$



Comparison to simplified problem

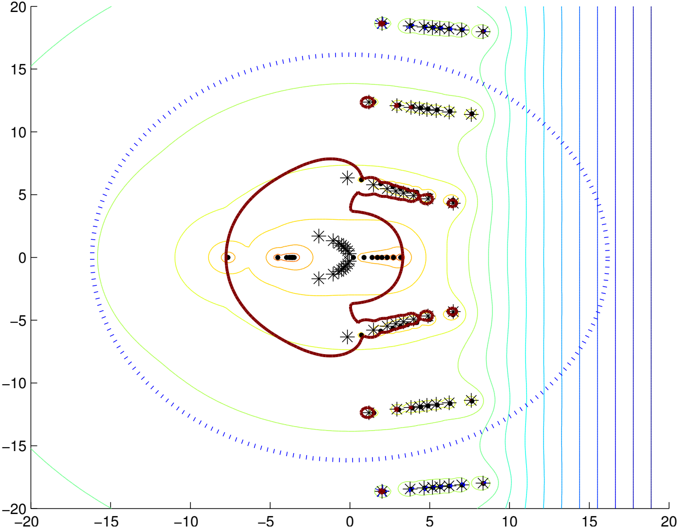
Bauer-Fike idea: apply a similarity!

$$T(z) = (e^z - 1)B + z^2 A - \alpha I$$

$$\begin{aligned}\tilde{T}(z) &= U^T T(z) U \\ &= (e^z - 1)D_B + z^2 I - \alpha E \\ &= D(z) - \alpha E\end{aligned}$$

$$G_i = \{z : |\beta_i(e^z - 1) + z^2| < \rho_i\}.$$

Gershgorin regions



A different comparison

Approximate $e^z - 1$ by a Chebyshev interpolant:

$$T(z) = (e^z - 1)B + z^2 A - \alpha I$$
$$\tilde{T}(z) = q(z)B + z^2 A - \alpha$$

$$T(z) = \tilde{T}(z) + r(z)B$$

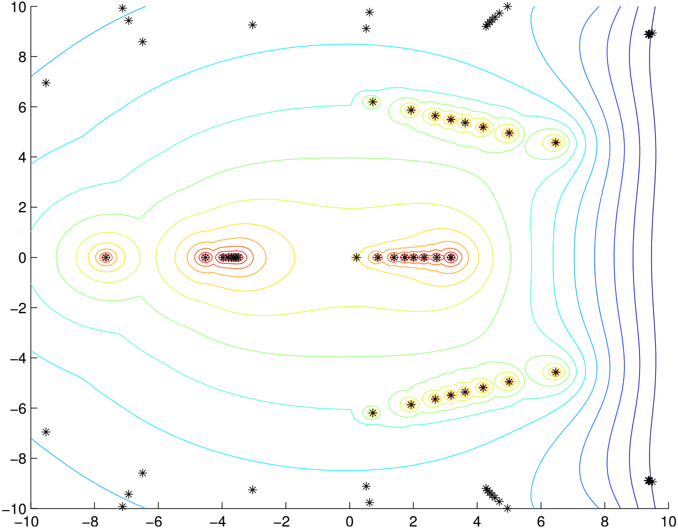
Linearize $\tilde{T}$ and transform both:

$$\tilde{T}(z) \mapsto D_C - zI$$
$$T(z) \mapsto D_C - zI + r(z)E$$

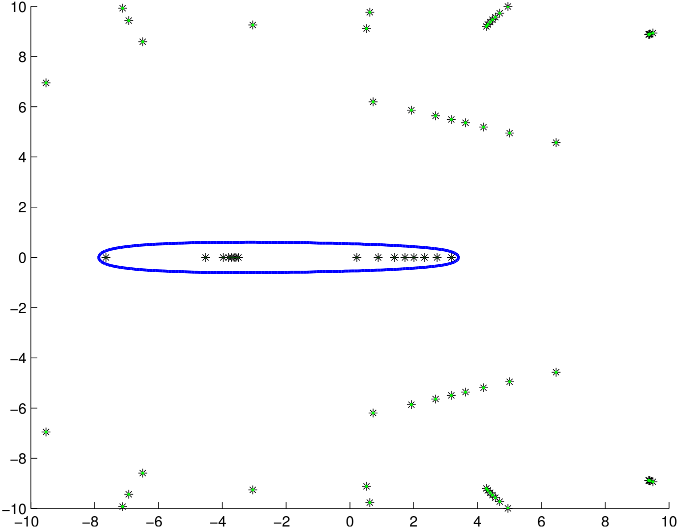
Restrict to $\Omega_\epsilon = \{z : |r(z)| < \epsilon\}$:

$$G_i \subset \hat{G}_i = \{z : |z - \mu_i| < \rho_i \epsilon\}, \qquad \rho_i = \sum_j |e_{ij}|$$

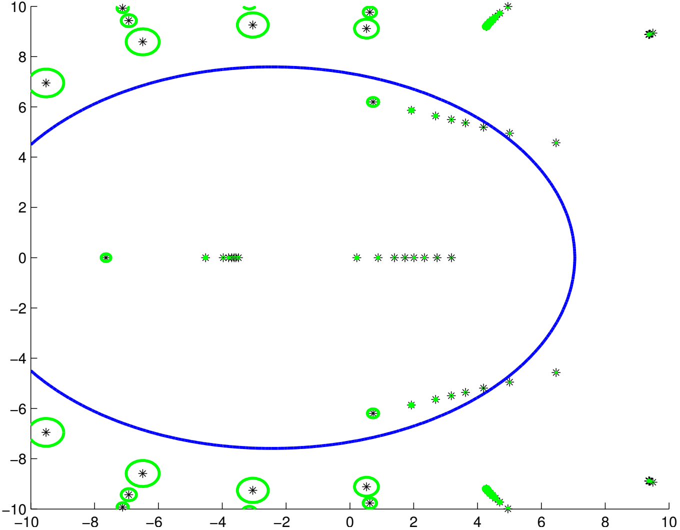
Spectrum of $\tilde{T}$



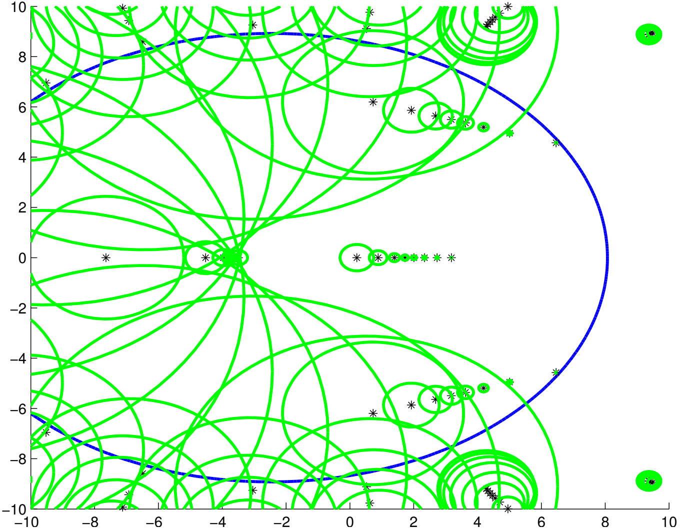
$\hat{G}_i$ for $\epsilon < 10^{-10}$



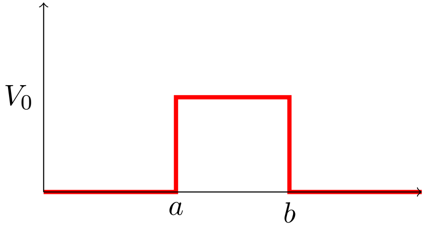
$\hat{G}_i$ for $\epsilon = 0.1$



$\hat{G}_i$ for $\epsilon = 1.6$



Example II: Resonance problem



$$\left(-\frac{d^2}{dx^2} + V - \lambda\right)\psi = 0 \qquad \text{on } (0, b),$$

$$\psi(0) = 0 \qquad \text{and} \qquad \psi'(b) = i\sqrt{\lambda}\psi(b),$$

Reduction via shooting

First-order form:

$$\frac{du}{dx} = \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix} u, \text{ where } u(x) \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}.$$

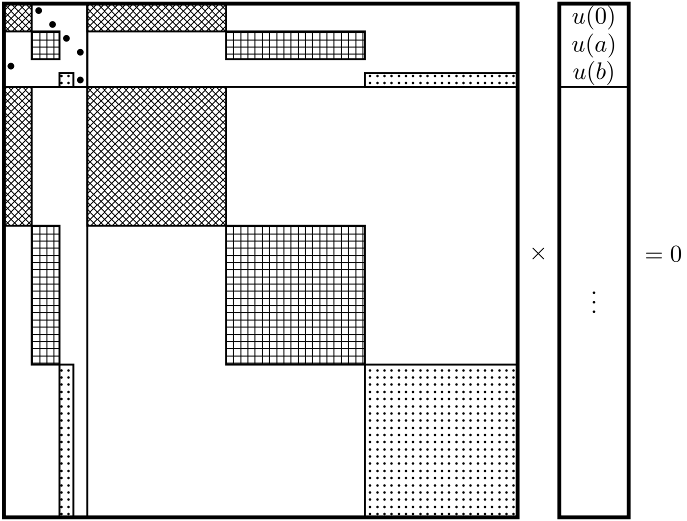
On region (c, d) where V is constant:

$$u(d) = R_{cd}(\lambda)u(c), \quad R_{cd}(\lambda) = \exp\left((d-c)\begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix}\right)$$

Reduce resonance problem to 6D NEP:

$$T(\lambda)u_{\text{all}} \equiv \begin{bmatrix} R_{0a}(\lambda) & -I & 0 \\ 0 & R_{ab}(\lambda) & -I \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 0 \\ -i\sqrt{\lambda} & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u(0) \\ u(a) \\ u(b) \end{bmatrix} = 0.$$

Expansion via rational approximation



Analyzing the expanded system

- $\hat{T}(z)$ is a Schur complement in $K - zM$
  - So $\Lambda(\hat{T})$ is easy to compute.
- Or: think $T(z)$ is a Schur complement in $K - zM + E(z)$
- Compare $\hat{T}(z)$ to $T(z)$ or compare $K - zM + E(z)$ to $K - zM$

Resonance approximation

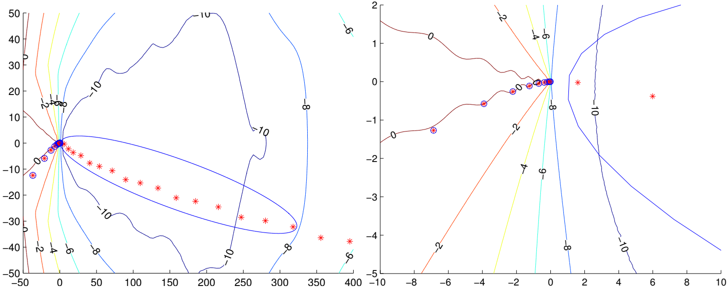


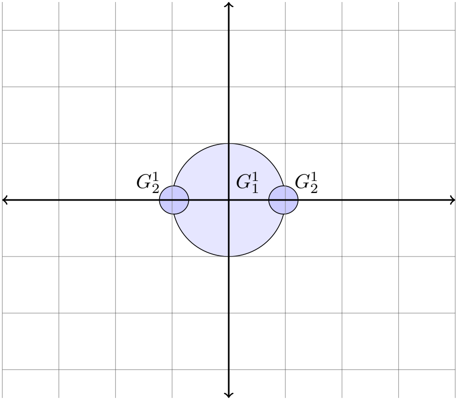
Figure : Circled eigenvalues satisfy $\|T(\lambda)\| > 10^{-8}$. Contour plots of $\log_{10}(\|T(z) - \hat{T}(z)\|)$ and an ellipse on which the smallest singular value of $T(z)$ is greater than 10^{-8}.

For more

Localization theorems for nonlinear eigenvalues.
David Bindel and Amanda Hood, SIMAX 34(4), 2013

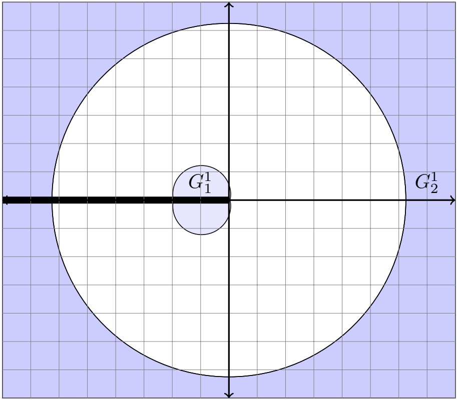
http://epubs.siam.org/doi/abs/10.1137/130913651

Example: Counting contributions



$$T(z) = \begin{bmatrix} z & 1 & 0 \\ 0 & z^2 - 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Domain boundaries



$$T(z) = \begin{bmatrix} z - 0.2\sqrt{z} + 1 & -1 \\ 0.4\sqrt{z} & 1 \end{bmatrix}$$

$$\Omega = \mathbb{C} - (-\infty, 0]$$

$$\det(D(z)) = (\sqrt{z} - 0.1 - i\sqrt{0.99})(\sqrt{z} - 0.1 + i\sqrt{0.99})$$

$$\det(T(z)) = (\sqrt{z} + 0.1 - i\sqrt{0.99})(\sqrt{z} + 0.1 + i\sqrt{0.99})$$

D has two eigenvalues in Ω;
T hides both eigenvalues behind a branch cut.