Music of the Microspheres
Eigenvalue problems from micro-gyro design

David Bindel
Department of Computer Science
Cornell University

UMCP NA seminar, 28 Jan 2014
A Favorite Application: MEMS

I’ve worked on this for a while:

- SUGAR (early 2000s) – SPICE for MEMS
- HiQLab (2006) – high-Q mechanical resonator device modeling
- AxFEM (2012) – solid-wave gyro device modeling

Goal today: an illustrative snapshot.
G. H. Bryan (1864–1928)

- Fellow of the Royal Society (1895)
- Stability in Aviation (1911)
- Thermodynamics, hydrodynamics

*Bryan was a friendly, kindly, very eccentric individual...*
(Obituary Notices of the FRS)

... if he sometimes seemed a colossal buffoon, he himself did not help matters by proclaiming that he did his best work under the influence of alcohol.

*(Williams, J.G., The University College of North Wales, 1884–1927)*
“On the beats in the vibrations of a revolving cylinder or bell”
by G. H. Bryan, 1890
Bryan’s Experiment Today
The Beat Goes On
A Small Application

Northrup-Grummond HRG
(developed c. 1965–early 1990s)
A Smaller Application (Cornell)
A Smaller Application (UMich, GA Tech, Irvine)
A Smaller Application!
Goal: Cheap, small (1mm) HRG

Collaborator roles:
- Basic design
- Fabrication
- Measurement

Our part:
- Detailed physics
- Fast software
- Sensitivity
- Design optimization
Foucault in Solid State
Rate Integrating Mode
Rate Integrating Mode
Consider free vibration consisting of two modes of oscillation:

\[ \ddot{\mathbf{q}} + \omega_0^2 \mathbf{q} = 0, \quad \mathbf{q}(t) \in \mathbb{R}^2. \]
A General Picture: Rotating Frame

Rate of rotation is $\Omega \ll \omega_0$:

$$\ddot{q} + 2\beta \Omega \mathbf{J} \dot{q} + \omega_0^2 q = 0,$$

$$\mathbf{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
A General Picture: Rotating Frame

\[ q_1(t) + q_2(t) \approx \cos(-\beta \Omega t) - \sin(-\beta \Omega t) \begin{bmatrix} q_1^0(t) \\ q_2^0(t) \end{bmatrix}. \]
An Uncritical FEA Approach

Why not do the obvious?

- Build 3D model with commercial FE
- Run modal analysis
The Perturbation Picture

Perturbations split degenerate modes:

- Coriolis forces (good)
- Imperfect fab (bad, but physical)
- Discretization error (non-physical)
Three Step Program

1. Perfect geometry, no rotation
2. Perfect geometry, rotation
3. Imperfect geometry
Step I: Perfect Geometry, No Rotation
Step I: Perfect Geometry, No Rotation

Free vibration problem in weak form

$$\forall \mathbf{w}, \quad b(\mathbf{w}, \ddot{\mathbf{u}}) + a(\mathbf{w}, \mathbf{u}) = 0.$$ 

where

$$b(\mathbf{w}, \mathbf{a}) = \int_{\mathcal{B}_0} \rho \mathbf{w} \cdot \mathbf{a} \, d\mathcal{B}_0,$$

$$a(\mathbf{w}, \mathbf{u}) = \int_{\mathcal{B}_0} \varepsilon(\mathbf{w}) : C : \varepsilon(\mathbf{u}) \, d\mathcal{B}_0,$$
Step I: Perfect Geometry, No Rotation

Free vibration problem in weak form

$$\forall w, \quad b(w, \ddot{u}) + a(w, u) = 0.$$ 

Symmetry: $Q$ any rotation or reflection

$$b(Qw, Qu) = b(w, u)$$
$$a(Qw, Qu) = a(w, u)$$

Decompose by invariant subspaces of $Q$ $\implies$ Fourier analysis
Fourier Expansion and Axisymmetric Shapes

Decompose into symmetric $u^c$ and antisymmetric $u^s$ in $y$:

$$u^c = \sum_{m=0}^{\infty} \Phi^c_m(\theta) u^c_m(r, z), \quad u^s = \sum_{m=0}^{\infty} \Phi^s_m(\theta) u^s_m(r, z)$$

where

$$\Phi^c_m(\theta) = \text{diag}(\cos(m\theta), \sin(m\theta), \cos(m\theta))$$
$$\Phi^s_m(\theta) = \text{diag}(-\sin(m\theta), \cos(m\theta), -\sin(m\theta)).$$

Modes involve only one azimuthal number $m$; degenerate for $m > 1$.

Preserve structure in FE: shape functions $N_j(r, z) \Phi^c,s_m(\theta)$
Finite element system: $M \ddot{u}^h + Ku^h = 0$

$$
K = \begin{bmatrix}
K^{cc}_0 & K^{ss}_1 & K^{cc}_1 & \cdots & K^{ss}_M \\
K^{ss}_1 & K^{cc}_1 & K^{ss}_2 & \cdots & K^{cc}_M \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
K^{ss}_M & K^{cc}_M & \cdots & \cdots & K^{cc}_M
\end{bmatrix}
$$

Mass has same structure.
Step II: Perfect Geometry, Rotation

Free vibration problem in weak form

\[ \forall w, \quad b(w, a) + a(w, u) = 0. \]

where

\[ a = \ddot{u} + 2\Omega \times \dot{u} + \Omega \times (\Omega \times x) + \dot{\Omega} \times x \]

Discretize by finite elements as before:

\[ M\ddot{u}^h + C\dot{u}^h + K\dot{u}^h = 0 \]

where \( C \) comes from Coriolis term \((2b(w, \Omega \times \dot{u}))\).
Block Structure of Finite Element Matrix

Discretize $2b(w, \Omega \times \dot{u})$:

$$C = \begin{bmatrix}
C_{00} & C_{01} \\
C_{10} & C_{11} & C_{12} \\
C_{21} & C_{22} & C_{23} \\
& & \ddots & \ddots & \ddots
\end{bmatrix}$$

Off-diagonal blocks come from cross-axis sensitivity:

$$\Omega = \Omega_z e_z + \Omega_{r\theta} = \begin{bmatrix} 0 \\ 0 \\ \Omega_z \end{bmatrix} + \begin{bmatrix} \cos(\theta)\Omega_x - \sin(\theta)\Omega_y \\ \sin(\theta)\Omega_x + \cos(\theta)\Omega_y \\ 0 \end{bmatrix}.$$  

Neglect cross-axis effects ($O(\Omega^2/\omega_0^2)$, like centrifugal effect).
Analysis in Ideal Case

Only need to mesh a 2D cross-section!

- Compute an operating mode $u_c$ for the non-rotating geometry.
- Compute associated modal mass and stiffness $m$ and $k$.
- Compute $g = b(u_c, e_z \times u_s)$.
- Model: motion is approximately $q_1 u_c + q_2 u_s$, and

$$m\ddot{q} + 2g\Omega J\dot{q} + kq = 0,$$
Step III: Imperfect Geometry
Representing the Perturbation

Map axisymmetric $B_0 \rightarrow$ real $B$:

$$ R \in B_0 \mapsto r = R + \epsilon \psi(R) \in B. $$

Write weak form in $B_0$ geometry:

$$ b(w, a) = \int_{B_0} \rho w \cdot a \ J \ dB_0, $$

$$ a(w, u) = \int_{B_0} \varepsilon(w) : C : \varepsilon(u) \ J \ dB_0, $$

where $J = \det(I + \epsilon F)$ with $F = \partial \psi / \partial R$. 
Decomposing $\psi$

Do Fourier decomposition of $\psi$, too! Consider case where

$$m = \text{only azimuthal number of } w$$
$$n = \text{only azimuthal number of } u$$
$$p = \text{only azimuthal number of } \psi$$

Then we have selection rules

$$a(w, u) = \begin{cases} 
O(\epsilon^k), & |m \pm n| = kp \\
0, & \text{otherwise}
\end{cases}$$

Similar picture for $b$. 
Decomposing $\psi$

- Over/under etch ($p = 0$)
- Mask misalignment ($p = 1$)
- Thickness variations ($p = 1$)
- Anisotropy of etching single-crystal Si ($p = 3$ or $p = 4$)
Block matrix structure

Ex: $p = 2$

$$K = \begin{bmatrix}
K_0 & \epsilon & \epsilon^2 & \epsilon^3 \\
\epsilon & K_1 & \epsilon & \epsilon^2 \\
\epsilon^2 & \epsilon & K_2 & \epsilon \\
\epsilon^2 & \epsilon & \epsilon & K_3 \\
\epsilon^3 & \epsilon^2 & \epsilon & K_4 \\
\epsilon^3 & \epsilon^2 & \epsilon & K_5 \\
& & & K_6 \\
& & & \ddots
\end{bmatrix}$$
Impact of Selection Rules

- Fast FEA: Can neglect some wave numbers / blocks
  - All assuming no accidental (near) degeneracies
  - First order: Only need diagonal blocks
  - Second order: Diagonal blocks plus “directly coupled”

- Also *qualitative* information
Qualitative Information

Operating wave number $m$, perturbation number $p$:

<table>
<thead>
<tr>
<th>$p = 2m$</th>
<th>frequencies split by $O(\epsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$kp = 2m$</td>
<td>frequencies split at most $O(\epsilon^2)$</td>
</tr>
<tr>
<td>$p \neq 2m$</td>
<td>frequencies change at $O(\epsilon^2)$, no split</td>
</tr>
<tr>
<td>$p = 1$</td>
<td>$O(\epsilon)$ cross-axis coupling.</td>
</tr>
<tr>
<td>$p = 2m \pm 1$</td>
<td></td>
</tr>
</tbody>
</table>

Note:

- $m = 2$ affected at first order by $p = 0$ and $p = 4$
  (and $O(\epsilon^2)$ split from $p = 1$ and $p = 2$).

- $m = 3$ affected at first order by $p = 0$ and $p = 6$
  (and $O(\epsilon^2)$ split from $p = 1$ and $p = 3$).
Mode Split for Rings: $\psi(r, \theta) = (\cos(2m\theta), 0)$.
Mode Split for Rings: $\psi(r, \theta) = (\cos(m\theta), 0)$. 

![Graph showing frequency shift as a function of perturbation amplitude/thickness for different values of $m$. The graph includes lines and markers for $m = 2a$, $m = 2b$, $m = 3a$, $m = 3b$, $m = 4a$, and $m = 4b$.](image-url)
Analyzing Imperfect Rings
Analyzing Imperfect Rings
Beyond Rings: AxFEM

- Mapped finite / spectral element formulation
- Low-order polynomials through thickness
- High-order polynomials along length
- Trig polynomials in $\theta$
- Agrees with results reported in literature
- Computes sensitivity to geometry, material parameters, etc.
Further Steps

Lots of possible directions:

- Symmetry breaking through damping?
- Integration with fabrication simulation?
- Joint optimization of geometry and fabrication?
Thank You

Yilmaz and Bindel
“Effects of Imperfections on Solid-Wave Gyroscope Dynamics”

Thanks to DARPA MRIG + Sunil Bhave and Laura Fegely.