Computer Aided Design of Micro-Electro-Mechanical Systems
From Eigenvalues to Devices

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The Computational Science & Engineering Picture

- Application
- Analysis
- Computation
A Favorite Application: MEMS

I’ve worked on this for a while:

- SUGAR (early 2000s) – SPICE for MEMS
- HiQLab (2006) – high-Q mechanical resonator device modeling
- AxFEM (2012) – solid-wave gyro device modeling

Goal today: two illustrative snapshots.
Outline

1. Resonant MEMS
2. Anchor losses and disk resonators
3. Elastic wave gyros
4. Conclusion
MEMS Basics

- Micro-Electro-Mechanical Systems
  - Chemical, fluid, thermal, optical (MECFTOMS?)
- Applications:
  - Sensors (inertial, chemical, pressure)
  - Ink jet printers, biolab chips
  - Radio devices: cell phones, inventory tags, pico radio
- Use integrated circuit (IC) fabrication technology
- Tiny, but still classical physics
Where are MEMS used?
My favorite applications
Why you should care, too!
The Mechanical Cell Phone

...and lots of mechanical sensors, too!
Ultimate Success

“Calling Dick Tracy!”

I'm On My Way
Computational Challenges

Devices are fun – but I’m not a device designer. Why am I in this?
Model System

\[ V_{in} \]

\[ V_{out} \]
The Circuit Designer View
Electromechanical Model

Balance laws (KCL and BLM):

\[
\frac{d}{dt} (C(u)V) + GV = I_{\text{external}}
\]

\[
Mu_{tt} + Ku - \nabla_u \left( \frac{1}{2} V^* C(u)V \right) = F_{\text{external}}
\]

Linearize about static equilibrium \((V_0, u_0)\):

\[
C(u_0) \delta V_t + G \delta V + (\nabla_u C(u_0) \cdot \delta u_t) V_0 = \delta I_{\text{external}}
\]

\[
M \delta u_{tt} + \tilde{K} \delta u + \nabla_u (V_0^* C(u_0) \delta V) = \delta F_{\text{external}}
\]

where

\[
\tilde{K} = K - \frac{1}{2} \frac{\partial^2}{\partial u^2} (V_0^* C(u_0)V_0)
\]
Electromechanical Model

Assume time-harmonic steady state, no external forces:

\[
\begin{bmatrix}
i \omega C + G & i \omega B \\
-B^T & \tilde{K} - \omega^2 M
\end{bmatrix}
\begin{bmatrix}
\delta \hat{V} \\
\delta \hat{u}
\end{bmatrix}
= \begin{bmatrix}
\delta \hat{I}_{\text{external}} \\
0
\end{bmatrix}
\]

Eliminate the mechanical terms:

\[
Y(\omega) \delta \hat{V} = \delta \hat{I}_{\text{external}}
\]

\[
Y(\omega) = i \omega C + G + i \omega H(\omega)
\]

\[
H(\omega) = B^T \left( \tilde{K} - \omega^2 M \right)^{-1} B
\]

Goal: Understand electromechanical piece \((i \omega H(\omega))\).

- As a function of geometry and operating point
- Preferably as a simple circuit
Damping and $Q$

Designers want high *quality of resonance* ($Q$)

- Dimensionless damping in a one-dof system

$$\frac{d^2 u}{dt^2} + Q^{-1} \frac{du}{dt} + u = F(t)$$

- For a resonant mode with frequency $\omega \in \mathbb{C}$:

$$Q := \frac{|\omega|}{2 \text{Im}(\omega)} = \frac{\text{Stored energy}}{\text{Energy loss per radian}}$$

To understand $Q$, we need damping models!
The Designer’s Dream

Reality is messy:
- Coupled physics
- ... some poorly understood (damping)
- ... subject to fabrication errors

Ideally, would like:
- Simple models for behavioral simulation
- Parameterized for design optimization
- Including all relevant physics
- With reasonably fast and accurate set-up

We aren’t there yet.
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Disk Resonator Simulations
Damping Mechanisms

Possible loss mechanisms:
- Fluid damping
- Material losses
- Thermoelastic damping
- Anchor loss

Model substrate as semi-infinite $\Rightarrow$ resonances!
Resonances in Physics
In bells of frost I heard the resonance die.
– Li Bai (translated by Vikram Seth)
Perfectly Matched Layers

- Complex coordinate transformation
- Generates a “perfectly matched” absorbing layer
- Idea works with general linear wave equations
  - Electromagnetics (Berengér, 1994)
  -Quantum mechanics – exterior complex scaling (Simon, 1979)
  - Elasticity in standard finite element framework (Basu and Chopra, 2003)
  - Works great for MEMS, too! (Bindel and Govindjee, 2005)
Anchor losses and disk resonators

Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate

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Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate
Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate

Re($\tilde{x}$)
Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate

Re($\tilde{x}$)
Anchor losses and disk resonators

Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate

Re($\tilde{x}$)
Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate

$\Re(\tilde{x})$
Matrices are *complex symmetric*
Goal: understand $H(\omega)$:

$$H(\omega) = B^T(K - \omega^2 M)^{-1} B$$

Look at
- Poles of $H$ (eigenvalues)
- Bode plots of $H$

Model reduction: Replace $H(\omega)$ by cheaper $\hat{H}(\omega)$. 
Approximation from Subspaces

A general recipe for large-scale numerical approximation:

1. A subspace $\mathcal{V}$ containing good approximations.
2. A criterion for “optimal” approximations in $\mathcal{V}$.

Basic building block for eigensolvers and model reduction!

Better subspaces, better criteria, better answers.
Variational Principles

- Variational form for complex symmetric eigenproblems:
  - Hermitian (Rayleigh quotient):
    \[ \rho(v) = \frac{v^* K v}{v^* M v} \]
  - Complex symmetric (modified Rayleigh quotient):
    \[ \theta(v) = \frac{v^T K v}{v^T M v} \]

- First-order accurate eigenvectors \( \implies \)
  Second-order accurate eigenvalues.
- Good for model reduction, too!
Disk Resonator Simulations
Axisymmetric model, bicubic, $\approx 10^4$ nodal points at convergence
Model Reduction Accuracy

Results from ROM (solid and dotted lines) nearly indistinguishable from full model (crosses)
Model Reduction Accuracy

Preserve structure → get twice the correct digits

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Variation in Quality of Resonance

Simulation and lab measurements vs. disk thickness

(Cornell University)

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Explanation of $Q$ Variation

![Graph showing the variation of $Q$ with film thickness (µm).](image-url)
Explanation of $Q$ Variation

Interaction of two nearby eigenmodes

- a = 1.51 µm
- b = 1.52 µm
- c = 1.53 µm
- d = 1.54 µm
- e = 1.55 µm
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Bryan’s Experiment

“On the beats in the vibrations of a revolving cylinder or bell”
by G. H. Bryan, 1890
A Small Application

Northrup-Grummond HRG
Current example: Micro-HRG / GOBLiT / OMG

- **Goal:** Cheap, small (1mm) HRG
- **Collaborator roles:**
  - Basic design
  - Fabrication
  - Measurement
- **Our part:**
  - Detailed physics
  - Fast software
  - Sensitivity
  - Design optimization
How It Works
How It Works
Goal state

We want to compute:

- Geometry
- Fundamental frequencies
- Angular gain (Bryan’s factor)
- Damping (thermoelastic, radiation, material)
- Sensitivities of everything
- Effects of symmetry breaking

For speed and accuracy: use structure!

- Axisymmetric geometry $\implies$ 3D to 2D via Fourier
- Perturbed geometry $\implies$ interactions for different wave numbers
Simple isotropic etch modeling fails – 1mm is *huge*!

Working on better simulator (reaction-diffusion).

For now, take idealized geometries on faith...
Full Dynamics
Elastic wave gyros

Essential Dynamics

Dynamics in 2D subspace of degenerate modes:

\[ (-\omega^2 m I + 2i\omega \Omega g J + k I) \mathbf{c} = 0 \]

Scaled gain \( g \) is \textit{Bryan’s factor}

\[ BF = \frac{\text{Angular rate of pattern relative to body}}{\text{Angular rate of vibrating body}} \]
If no parameters in the world were very large or very small, science would reduce to an exhaustive list of everything.

– Nick Trefethen
Write displacement fields as Fourier series:

\[
u = \sum_{m=0}^{\infty} \begin{bmatrix} u_{mr}(r, z) \cos(m\theta) \\ u_{m\theta}(r, z) \sin(m\theta) \\ u_{mz}(r, z) \cos(m\theta) \end{bmatrix} + \begin{bmatrix} -u'_{mr}(r, z) \sin(m\theta) \\ u'_{m\theta}(r, z) \cos(m\theta) \\ -u'_{mz}(r, z) \sin(m\theta) \end{bmatrix}\]

- Works whenever \textit{geometry} is axisymmetric
- Treat non-axisymmetric geometries as mapped axisymmetric
  - Now \textit{coefficients} in PDEs are non-axisymmetric
- Problems with different \textit{m} decouple if \textit{everything} axisymmetric
Perfect axisymmetry:

\[
\begin{bmatrix}
K_{11} & K_{22} & \cdots \\
K_{22} & K_{33} & \cdots \\
\cdots & \cdots & \ddots
\end{bmatrix} - \omega^2
\begin{bmatrix}
M_{11} & M_{22} \\
M_{22} & M_{33} \\
\cdots & \cdots
\end{bmatrix}
\]
Broken symmetry (via coefficients):

\[
\begin{bmatrix}
K_{11} & \epsilon & \epsilon & \epsilon \\
\epsilon & K_{22} & \epsilon & \epsilon \\
\epsilon & \epsilon & K_{33} & \epsilon \\
\epsilon & \epsilon & \epsilon & \ddots
\end{bmatrix}
- \omega^2
\begin{bmatrix}
M_{11} & \epsilon & \epsilon & \epsilon \\
\epsilon & M_{22} & \epsilon & \epsilon \\
\epsilon & \epsilon & M_{33} & \epsilon \\
\epsilon & \epsilon & \epsilon & \ddots
\end{bmatrix}
\]
Perturbing Fourier

Modes “near” azimuthal number $m = \text{nonlinear eigenvalues}$

$$\left( K_{mm} - \omega^2 M_{mm} + E_{mm}(\omega) \right) u = 0.$$ 

Need:
- Control on $E_{mm}$
  - Depends on frequency spacing
  - Depends on Fourier analysis of perturbation
- Perturbation theory for nonlinearly perturbed eigenproblems
  - For self-adjoint case, results similar to Lehmann intervals

First-order estimate:
$$\left( K_{mm} - \omega_0^2 M_{mm} \right) u_0 = 0; \text{ then}$$

$$\delta(\omega^2) = \frac{u_0^T E_{mm}(\omega_0) u_0}{u_0^T M_{mm} u_0}.$$
Perturbation and Radiation

Incorporating numerical radiation BCs gives:

$$(K - \omega^2 M + G(\omega))u = 0.$$  

Perturbation approach: ignore $G$ to get $(\omega_0, u_0)$. Then

$$\delta(\omega^2) = \frac{u_0^T G(\omega_0) u_0}{u_0^T M_{mm} u_0}.$$  

Works when BC has small influence (coefficients aren’t small).

Also an approach to understanding sensitivity to BC!  
... explains why PML works okay despite being inappropriate?
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Conclusions

The difference between art and science is that science is what we understand well enough to explain to a computer. Art is everything else.

Donald Knuth

The purpose of computing is insight, not numbers.

Richard Hamming

Collaborators:
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- HRG: S. Bhave, L. Fegely, E. Yilmaz

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