Some perturbation theorems for nonlinear eigenvalue problems

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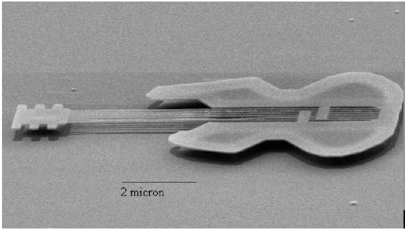
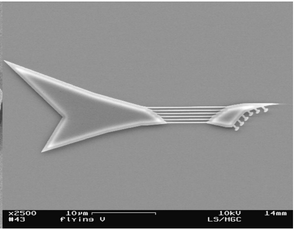
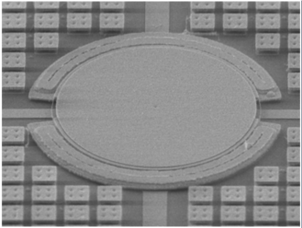
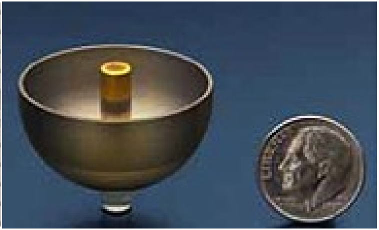
Why nonlinear eigenvalue problems?









The general setting

Nonlinear eigenvalue problem:

$$T(\lambda)v = 0, \quad v \neq 0.$$

where

- $T : \Omega \to \mathbb{C}^{n \times n}$ analytic on simply connected $\Omega \subset \mathbb{C}$
- $\det(T) \not\equiv 0$ (i.e. T is regular)

Write the set of nonlinear eigenvalues as $\Lambda(T)$.

Source: transform methods on almost anything with damping!
For many examples, see:

- NLEVP collection
- Survey by Mehrmann and Voss

Quadratic problems

Example: Damped free vibrations of a mechanical system

$$Mu'' + Bu' + Ku = 0.$$

Laplace transform:

$$(s^2 M + sB + K)U = 0.$$

Approach directly or convert to first order:

$$\begin{aligned} Bv + Ku &= sMv \\ v &= su \end{aligned}$$

Polynomial problems

More general is *polynomial* eigenvalue problem:

$$T(\lambda)v = 0, \quad T(z) \equiv z^d I + z^{d-1} A_d + \ldots + zA_1 + A_0$$

Common approach: define $u_j = \lambda^j v$, and solve

$$\begin{bmatrix} -A_{d-1} & -A_{d-2} & \ldots & -A_1 & -A_0 \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{bmatrix} \begin{bmatrix} u_{d-1} \\ u_{d-2} \\ \vdots \\ u_1 \\ v \end{bmatrix} = \lambda \begin{bmatrix} u_{d-1} \\ u_{d-2} \\ \vdots \\ u_1 \\ v \end{bmatrix}$$

This is one of many possible *linearizations*.
Can do something similar with rational problems.

A special rational problem

Consider the eigenvalue equation

$$\begin{bmatrix} A-\lambda I & B \\ C & D-\lambda I \end{bmatrix} \begin{bmatrix} v \\ \tilde{v} \end{bmatrix} = 0.$$

If $\lambda \notin \Lambda(D)$, partial Gaussian elimination yields $T(\lambda)v = 0$, where

$$T(z) = A - zI - B(D - zI)^{-1}C.$$

This is a *spectral Schur complement* problem.

(c.f. Feschbach, Lifschitz, Grushin).

Solving general NEPs

$$T(\lambda)x = 0, \quad x \neq 0, \quad T : \Omega \to \mathbb{C}^{n \times n} \text{ analytic}$$

Computational approaches:

- Local polynomial / rational approximation of T
- Methods based on contour integration

Either way, we want:

- A starting point (expansion point, contour)
- Error estimates for the results

Perturbation and localization

Many uses for perturbation theory in linear case:

- Backward error analysis (first-order theory, pseudospectra)
- Crude bounds for choosing algorithm parameters (Gerschgorin)
- Crude bounds for stability testing (Gerschgorin)
- Reasoning about dynamics (pseudospectra)

Want the same theory for nonlinear problems!

First-order perturbation theory

Small, analytic E, consider

$$\hat{T} = T + E$$

Given a simple eigentriple (λ, u, w^*) of T:

$$T(\lambda)u = 0, \quad w^*T(\lambda) = 0.$$

First-order perturbation theory gives:

$$\delta\lambda = -\frac{w^*E(\lambda)u}{w^*T'(\lambda)u}$$

Great! What about large perturbations, multiple eigenvalues, ...?

Beyond first order

Suppose

- $T, E : \Omega \to \mathbb{C}^{n \times n}$ analytic
- $\Gamma \subset \Omega$ a simple contour
- $T(z) + sE(z)$ nonsingular, all $s \in [0, 1]$, $z \in \Gamma$.

Then T and $T + E$ have the same number of eigenvalues inside Γ.

Proof:
The winding number of $\det(T + sE)$ stays continuous for $0 \leq s \leq 1$.

A general recipe

Analyticity of T and E +
Matrix nonsingularity test for $T + sE$ =

Inclusion region for $\Lambda(T + E)$ +
Eigenvalue counts for connected components of region

Matrix Rouché

$\|T(z)^{-1}E(z)\| < 1$ on $\Gamma \implies$ same eigenvalue count in Γ

Proof:
$\|T(z)^{-1}E(z)\| < 1 \implies T(z) + sE(z)$ invertible for $0 \leq s \leq 1$.

(Gohberg and Sigal proved a more general version in 1971.)

Nonlinear pseudospectra

Define the nonlinear ϵ-pseudospectrum as

$$\Lambda_\epsilon(T) = \{z \in \Omega : \|T(z)^{-1}\| > \epsilon^{-1}\}$$

Let $\mathcal{E} = \{E : \Omega \to \mathbb{C}^{n \times n}$ s.t. E analytic, $\max_{z \in \Omega} \|E(z)\| < \epsilon\}$. Then

$$\Lambda_\epsilon(T) = \bigcup_{E \in \mathcal{E}} \Lambda(T + E).$$

If $\mathcal{E}_0 = \{E \in \mathbb{C}^{n \times n} : \|E_0\| < \epsilon\}$, we may also write

$$\Lambda_\epsilon(T) = \bigcup_{E_0 \in \mathcal{E}_0} \Lambda(T + E_0).$$

Nonlinear pseudospectra and backward error

Suppose $\hat{\lambda}, \hat{v}$ an approximate eigenpair with $\|\hat{v}\| = 1$,

$$T(\hat{\lambda})\hat{v} = r, \quad \|r\| \text{ small.}$$

Then $\hat{\lambda} \in \Lambda_{\|r\|}(T)$, since $\left(T(\hat{\lambda}) - r\hat{v}^*\right) v = 0$

Nonlinear pseudospectra and dynamics

Suppose $\Psi : [0, \infty) \to \mathbb{C}^{N \times N}$, let

$$R(z) \equiv \int_0^\infty e^{-zt} \Psi(t) \, dt.$$

Ψ bounded $\implies$ $R(z)$ defined in RHP and for any $\epsilon > 0$,

$$\sup_{t>0} \|\Psi(t)\| \geq \frac{\alpha_\epsilon}{\epsilon},$$

where

$$\alpha_\epsilon \equiv \sup_{\|R(\lambda_\epsilon)\| > \epsilon^{-1}} \operatorname{Re}(\lambda_\epsilon)$$

(Similar proof to that for linear pseudospectra.)

Pseudospectral counting

Let T, E analytic on Ω and define:

$$\Omega_\epsilon \equiv \{z \in \Omega : \|E(z)\| < \epsilon\}.$$

Then

$$\Lambda(T) \cap \Omega_\epsilon \subset \Lambda_\epsilon(T + E)$$

Also, if

- $\mathcal{U} \subset \Lambda_\epsilon(T + E)$ a connected component.
- $\bar{\mathcal{U}} \subset \Omega_\epsilon$.

then $\mathcal{U}$ contains the same number of eigenvalues of T and $T + E$, of which there must be at least one.

Weakly coupled problems

$$T(z) = \begin{bmatrix} L_1(z) & H(z) \\ G(z) & L_2(z) \end{bmatrix}$$

is analytic over Ω, and

$$\|G(z)\| \leq \gamma, \quad \|H(z)\| \leq \eta, \quad \Lambda_{\delta_1}(L_1) \cap \Lambda_{\delta_2}(L_2) = \emptyset.$$

Assume $\gamma\eta < \delta_1\delta_2$, boundary of $\Lambda_{\delta_1}(L_1)$ is strictly inside Ω. Then

1. $\Lambda(T) \subset \Lambda_{\delta_1}(L_1) \cup \Lambda_{\delta_2}(L_2)$
2. T and L_1 have same eigenvalue counts in $\Lambda_{\delta_1}(L_1)$
3. For $\lambda \in \Lambda_{\delta_1}(L_1)$, eigenvector v satisfies $\|v_2\|/\|v_1\| < \gamma/\delta_1$.
4. For $\lambda \in \Lambda_{\delta_2}(L_2)$, eigenvector v satisfies $\|v_2\|/\|v_1\| > \gamma/\delta_2$.

Linear problems, nonlinear perturbations

Perturb *linear* problem with E analytic, "small" on Ω:

$$T(z) = A - zB + E(z).$$

Many linear perturbation theorems still hold!

Nonlinear perturbations + pseudospectra

$$T(z) = A - zI + E(z)$$

and suppose $\|E\| < \epsilon$ on Ω.

If $\mathcal{U}$ a connected component of $\Lambda_\epsilon(A)$, $\bar{\mathcal{U}} \subset \Omega$, then

- A and T have the same eigenvalue counts in $\mathcal{U}$.
- The eigenvalue count in $\mathcal{U}$ is at least one.

Nonlinear Gerschgorin

For D diagonal, consider

$$T(z) = D - zI + E(z)$$

such that

$$\sum_{j=1}^{n} |e_{ij}(z)| \leq \rho_i$$

Then

- $\Lambda(T) \subset \bigcup_{i=1}^{n} G_i$ where $G_i = B_{\rho_i}(d_{ii})$
- $\mathcal{U} = \bigcup_{i \in \mathcal{I}} G_i$ a connected component, $\bar{\mathcal{U}} \subset \Omega$
  $\implies \mathcal{U}$ contains $|\mathcal{I}|$ eigenvalues.

Nonlinear Bauer-Fike bound

Suppose $|E(z)| \leq F$ componentwise on Ω,

$$T(z) = A - zI + E(z).$$

and A has eigentriples (λ_i, v_i, w_i^*). Then

$$\Lambda(T) \subset \bigcup_{i=1}^{n} B_{\phi_i}(\lambda_i)$$

where $\phi_i = n\|F\|_2 \sec(\theta_i)$ and

$$\sec(\theta_i) = \frac{\|w_i\|\|v_i\|}{|w_i^* v_i|}.$$

Can also count within connected components.

Application: Delay-differential equation

From NLEVP collection

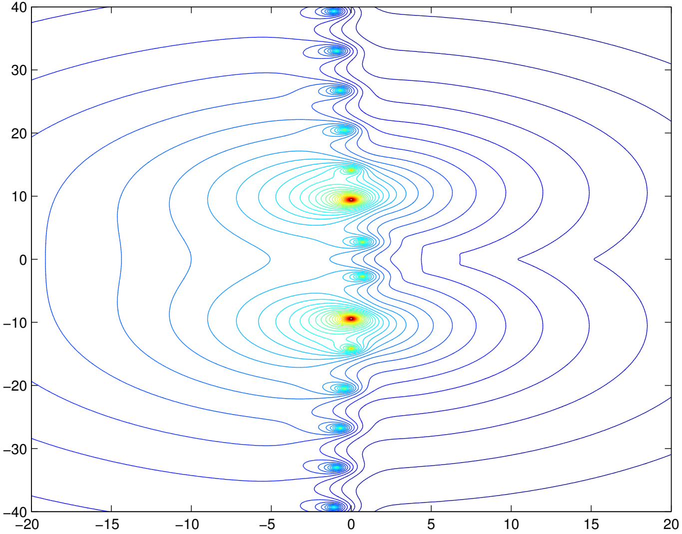
$$T(\lambda) = A_0 - \lambda I + A_1 \exp(-\lambda)$$

Corresponding to

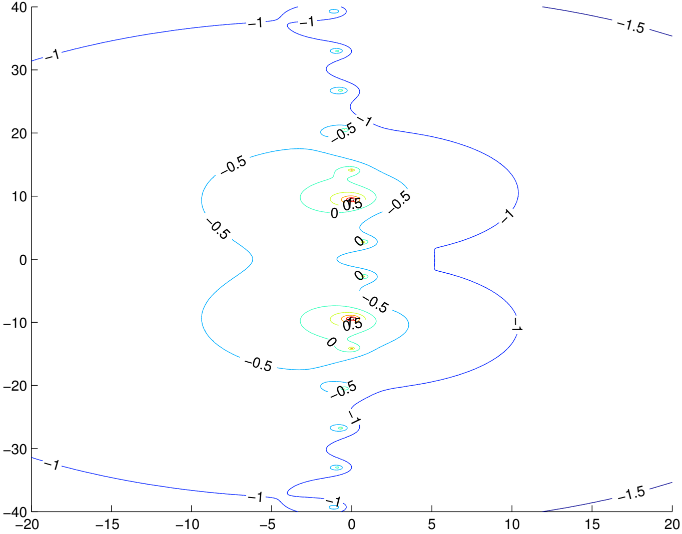
$$u'(t) = A_0 u(t) + A_1 u(t-1)$$

Double non-semisimple eigenvalue $\lambda = 3\pi i$.

Pseudospectral plot



Pseudospectral plot



Gerschgorin applied

Consider

$$V^{-1}T(\lambda)V = D - \lambda I + \tilde{A}_1 \exp(-\lambda)$$

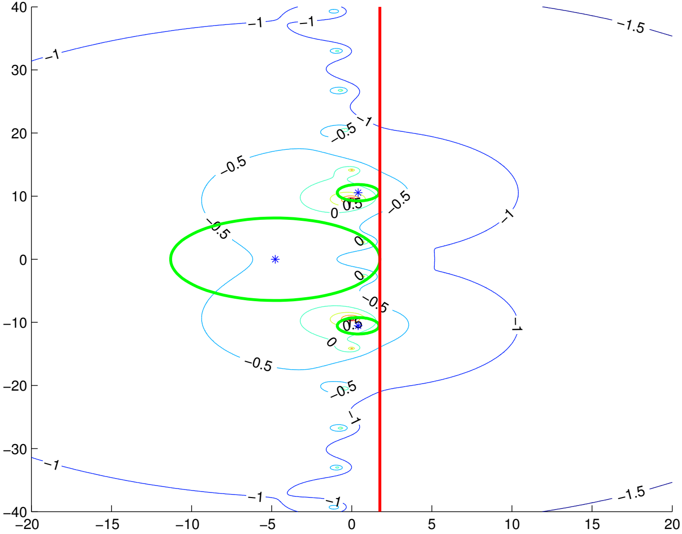
Apply Gerschgorin-like bound

$$\Lambda(T) \subset \bigcup_{i=1}^{3} B_{\rho_i}(d_{ii}) \cup \{|\exp(-\lambda)| > \exp(-\sigma)\}$$

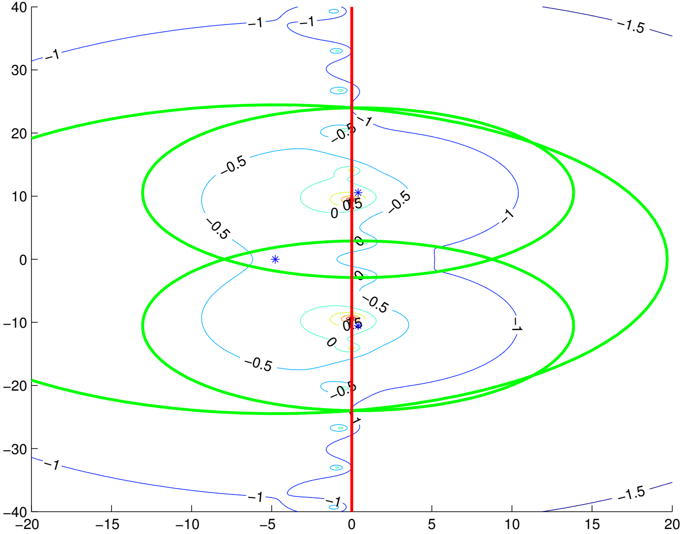
where

$$\rho_i = \exp(-\sigma) \left(\sum_j (\tilde{A}_1)_{ij} \right)^{\alpha} \left(\sum_j (\tilde{A}_1)_{ji} \right)^{1-\alpha}$$

Example: Bounding the spectral abscissa



Example: Imaginary part of unstable eigenvalues



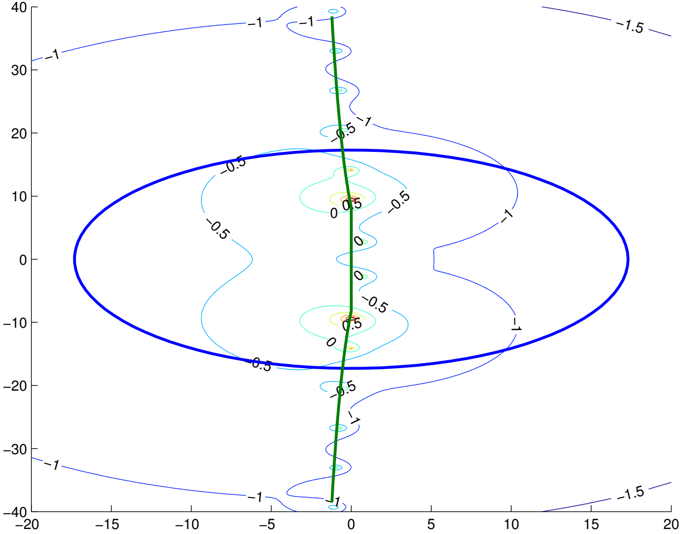
Switching terms

Consider

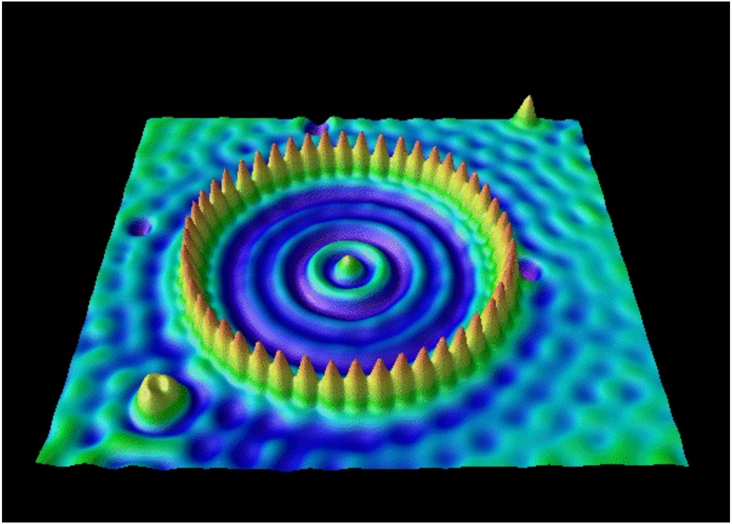
$$V^{-1}T(\lambda)V = D\exp(-\lambda) - \lambda + \tilde{A}_0$$

Gerschgorin-like argument now bounds spectrum from left!

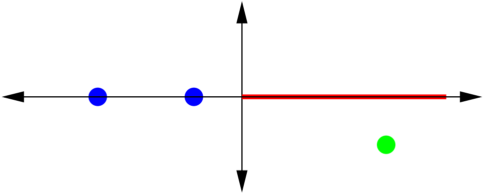
Example: Bounding spectrum from the left



Schrödinger resonances

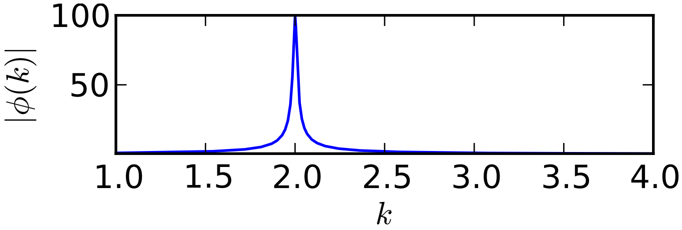


Spectra and scattering



Spectrum for $H = -\Delta + V$, $\mathrm{supp}(V)$ compact.

Resonances and scattering



For $\mathrm{supp}(V) \subset \Omega$, consider a scattering experiment:

$$(H - k^2)\psi = f \text{ on } \Omega$$
$$(\partial_n - B(k))\psi = 0 \text{ on } \partial\Omega$$

See resonance peaks (Breit-Wigner):

$$\phi(k) \equiv w^*\psi \approx C(k - k^*)^{-1}.$$

1D resonances: a quadratic eigenvalue problem

$$\left(-\frac{d^2}{dx^2} + V(x) - k^2\right)\psi = 0, \quad x \in (a, b)$$
$$\left(\frac{d}{dx} - ik\right)\psi = 0, \quad x = b$$
$$\left(\frac{d}{dx} + ik\right)\psi = 0, \quad x = a$$

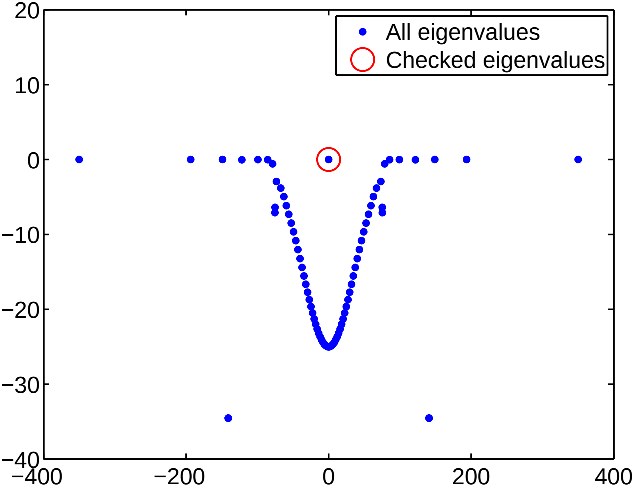
Look for nontrivial solutions:

- $\mathrm{Im}(k) > 0$: Bound states
- $\mathrm{Im}(k) < 0$: Resonances

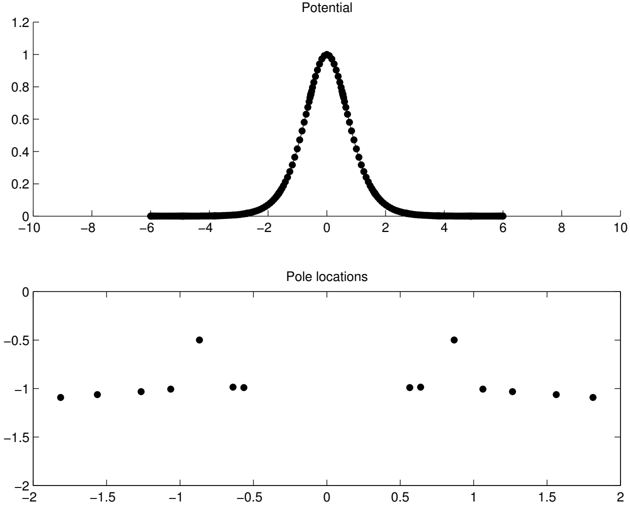
See:

`http://www.cs.cornell.edu/~bindel/sw/matscat/`

Is it that easy?



Is it that easy?



Sensitivity for resonances

Resonance solutions are stationary points with respect to ψ of

$$\begin{aligned}\Phi(\psi, k) &= \int_\Omega \psi \left[-\nabla^2\psi + (V - k^2)\psi\right] d\Omega - \int_{\partial\Omega} \psi \left(\frac{\partial\psi}{\partial n} - B(k)\psi \right) d\Gamma \\ &= \int_\Omega \left[(\nabla\psi)^T(\nabla\psi) + \psi(V - k^2)\psi\right] d\Omega - \int_{\partial\Omega} \psi B(k)\psi \, d\Gamma\end{aligned}$$

If (ψ, k) a resonance pair, then $\Phi(\psi, k) = 0$ and $D_\psi\Phi(\psi, k) = 0$.

Potential perturbations

If (ψ, k) a resonance pair, then $\Phi(\psi, k) = 0$ and $D_\psi \Phi(\psi, k) = 0$.

Consider perturbed V:

$$\delta\Phi = D_\psi \Phi \cdot \delta\psi + D_V \Phi \cdot \delta V + D_k \Phi \cdot \delta k = 0$$

Use $D_\psi \Phi \cdot \delta\psi = 0$:

$$\delta k = -\frac{D_V \Phi \cdot \delta V}{D_k \Phi}$$

Perturbation worked out

So look at how perturbations δV change k:

$$\delta k = \frac{\int_\Omega \delta V \psi^2}{2k \int_\Omega \psi^2 - \int_\Gamma \psi B'(k)\psi}$$

Can also write in terms of a residual for ψ as a solution for the potential $V + \delta V$:

$$\delta k = \frac{\int_\Omega \psi(-\Delta + (V + \delta V) - k^2)\psi}{2k \int_\Omega \psi^2 - \int_\Gamma \psi B'(k)\psi}.$$

Backward error analysis in MatScat

1. Compute approximate solution $(\hat{\psi}, \hat{k})$.
2. Map $\hat{\psi}$ to high-resolution quadrature grid to evaluate

$$\delta k = \frac{\int_\Omega \hat{\psi}(-\Delta + V - \hat{k}^2)\hat{\psi}}{2\hat{k}\int_\Omega \hat{\psi}^2 - \int_\Gamma \hat{\psi} B'(\hat{k})\hat{\psi}}.$$

3. If δk large, discard $\hat{k}$; otherwise, accept $k \approx \hat{k} + \delta k$.

Nonlinear vs linear eigenproblems

Can also compute resonances by

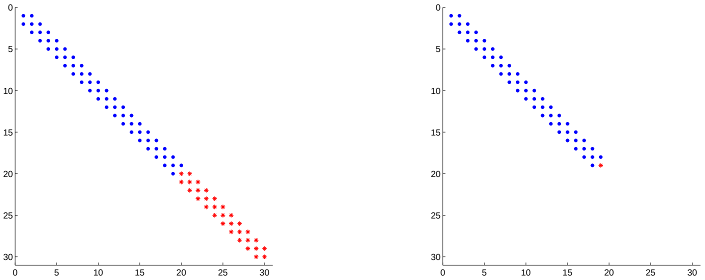
- Adding a complex absorbing potential
- Complex scaling methods
- Artificial dampers

Both result in complex-symmetric ordinary eigenproblems:

$$(K_{ext} - k^2 M_{ext})\psi_{ext} = \left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - k^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right) \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0$$

where ψ_2 correspond to extra variables (outside Ω).

Spectral Schur complement



Eliminate "extra" variables ψ_2 to get

$$\hat{T}(k)\psi_1 = \left(K_{11} - k^2 M_{11} - \hat{C}(k)\right)\psi_1 = 0$$

where

$$\hat{C}(k) = (K_{12} - k^2 M_{12})(K_{22} - k^2 M_{22})^{-1}(K_{21} - k^2 M_{21})$$

Apples to oranges?

$$\begin{aligned} T(k)\psi &= (K - k^2 M - C(k))\psi = 0 && \text{(exact DtN map)} \\ \hat{T}(\hat{k})\hat{\psi} &= (K - \hat{k}^2 M - \hat{C}(\hat{k}))\hat{\psi} = 0 && \text{(spectral Schur complement)} \end{aligned}$$

Two ideas:

- Perturbation theory for NEP for local refinement
- Complex analysis to get more global analysis

Aside on spectral Schur complement

Inverse of a Schur complement is a submatrix of an inverse:

$$(K_{ext} - z^2 M_{ext})^{-1} = \begin{bmatrix} \hat{T}(z)^{-1} & * \\ * & * \end{bmatrix}$$

So for reasonable norms,

$$\|\hat{T}(z)^{-1}\| \leq \|(K_{ext} - z^2 M_{ext})^{-1}\|.$$

Or

$$\Lambda_\epsilon(\hat{T}) \subset \Lambda_\epsilon(K_{ext}, M_{ext}),$$

$$\Lambda_\epsilon(\hat{T}) \equiv \{z : \|\hat{A}(z)^{-1}\| > \epsilon^{-1}\}$$
$$\Lambda_\epsilon(K_{ext}, M_{ext}) \equiv \{z : \|(K_{ext} - z^2 M_{ext})^{-1}\| > \epsilon^{-1}\}$$

Nonlinear bounds from linear pseudospectra

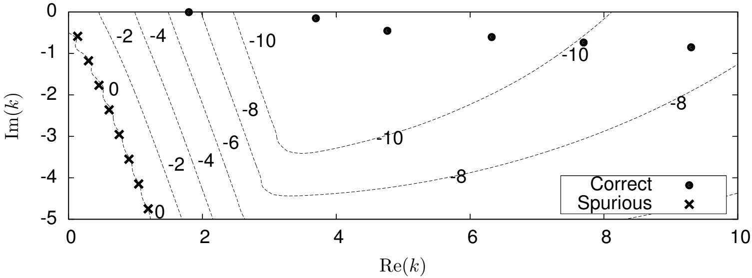
Recall:

$$T(k)\psi = (K - k^2 M - C(k))\psi = 0 \quad \text{(exact DtN map)}$$
$$\hat{A}(\hat{k})\hat{\psi} = (K - \hat{k}^2 M - \hat{C}(\hat{k}))\hat{\psi} = 0 \quad \text{(spectral Schur complement)}$$

Let $\Omega_\epsilon = \{z \in \mathbb{C} : \|C(z) - \hat{C}(z)\| < \epsilon\}$. Then:

$$\Lambda(T) \cap \Omega_\epsilon \subset \Lambda_\epsilon(\hat{T}) \subset \Lambda_\epsilon(K_{\text{ext}}, M_{\text{ext}})$$

Assessing approximate resonances



To get axisymmetric resonances in corral model, compute:

- Eigenvalues of a complex-scaled problem
- Residuals in nonlinear eigenproblem
- $\log_{10} \|T(k) - \hat{T}(k)\|$

Conclusion

- Nonlinear eigenvalue problems are as natural as linear problems
- Linear perturbation theorems with complex analytic proofs apply
- "Perturbation Theorems for Nonlinear Eigenvalue Problems"
  David Bindel and Amanda Hood