From Networks to Numerical Linear Algebra

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Example: Opinions in Networks
Basic model

Extension of DeGroot model due to Friedkin and Johnsen:

- Directed graph with nodes $1, \ldots, n$, weights $w_{ij}$
- Each node has two quantities:
  - Fixed internal opinion $s_i \in \mathbb{R}$
  - Variable expressed opinion $z_i \in \mathbb{R}$
- Update equation:

\[
Z_i^{\text{new}} \leftarrow \frac{s_i + \sum_{j \in N(i)} w_{ij} Z_j^{\text{old}}}{1 + \sum_{j \in N(i)} w_{ij}}
\]
Example: Opinions in Networks

\[ W = \begin{bmatrix} 0 & 0 & w_{AC} & 0 & 0 & w_{AP} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w_{CA} & 0 & 0 & 0 & 0 & w_{BP} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Matrix reformulation

Scalar form:

\[ Z_{i}^{\text{new}} \leftarrow \frac{S_{i} + \sum_{j \in N(i)} W_{ij} Z_{j}^{\text{old}}}{1 + \sum_{j \in N(i)} W_{ij}} \]

Matrix form:

\[(D + I)Z^{\text{new}} = s + WZ^{\text{old}}\]

This is Jacobi iteration! Converges to solution of

\[(L + I)x = s\]

where \(L = D - W\) is the (directed) graph Laplacian:

\[ L_{ij} = \begin{cases} -W_{ij}, & i \neq j \\ \sum_{k \in N(i)} W_{ik}, & i = j \end{cases} \]
Another perspective

Carol is “pulled” to:

- To be true to her personal beliefs \((z_C - s_C \text{ small})\)
- To agree with Alice \((z_C - z_A \text{ small})\)
- To agree with Paul \((z_C - z_B \text{ small})\)

She is unhappy to the extent that she cannot reconcile these.
Define local cost function:

\[ c_i = \frac{1}{2} \left( (z_i - s_i)^2 + \sum_{j \in N(i)} w_{ij} (z_i - z_j)^2 \right) \]

Node \( i \) chooses opinion to optimize \( z_i \):

\[ \frac{\partial c_i}{\partial z_i} = (z_i - s_i) + \sum_{j \in N(i)} w_{ij} (z_i - z_j) = 0 \]

Nash equilibrium satisfies \((L + I)x = s\).
Nash equilibrium vs social optimum

Define the *social cost*

\[
c(z) = \sum_i c_i(z) = \frac{1}{2} \left( z^T (A + I) z - 2 z^T s + s^T s \right).
\]

where \( A = L + L^T \)

- Nash equilibrium: Node \( i \) chooses \( x_i \) to minimize \( c_i \).
- Social optimum: Choose \( y \) globally to minimize \( c(y) \)

Equations for social optimum: \((A + I)y = s\).
Price of anarchy

The price of anarchy is

$$\text{PoA}(s) = \frac{c(y)}{c(x)} = \frac{s^T Bs}{s^T Cs}$$

where

\[
B = (A + I)^{-1} - I + (A + I)^{-1} A (A + I)^{-1} \\
C = ((L + I)^{-1} - I)^T ((L + I)^{-1} - I) + (L + I)^{-T} A (L + I)^{-1}
\]

Undirected case: \(L\) symmetric, \(B = p(L), C = q(L)\), and

\[
\max_{s \neq 0} \frac{s^T Bs}{s^T Cs} = \max_{\lambda \in \Lambda(L)} \frac{p(\lambda)}{q(\lambda)} \leq \max_{t \geq 0} \frac{p(t)}{q(t)} = \frac{9}{8}.
\]

Directed graphs: still an eigenvalue problem.
Mean opinion and influence

Let $e$ be the vector of all ones. Mean opinion at Nash is:

$$\bar{x} = \frac{1}{n} e^T x = \frac{1}{n} e^T (L + I)^{-1} s = f^T s$$

The influence vector

$$f = (L + I)^{-T} e$$

tells how much each node influences the mean opinion.
Influence in the model network

Bold lines are weight 2, regular are weight 1:

$$f = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{2} & 1 & \frac{19}{6} \end{bmatrix}^T$$
Distribution of influence

The *uniform influence* case $f = e/n$ occurs when

$$L^T e = 0,$$

i.e. graph is *Eulerian* (in-degree weight = out-degree weight). Uniform influence always true for socially optimal opinion!

In general, max influence is

$$\max_i f_i = \| (L + I)^{-1} \|_1$$

Note: $\| (L + I)^{-1} \|_\infty = 1$. What about other norms?
Variance of opinion

Assume $s$ is normalized to mean zero. Variance in intrinsic opinion:

$$\text{Var}[s] = \frac{1}{n} s^T s$$

What about the variance in the expressed opinion at Nash?

$$\frac{\text{Var}[x]}{\text{Var}[s]} = \frac{s^T (L + I)^{-1}(I - ee^T/n)(L + I)^{-1}s}{s^T s}$$

So if $s$ is not identically zero, then

$$\frac{\sigma_x}{\sigma_s} \leq \| (I - ee^T/n)(L + I)^{-1} \|_2 \leq \sqrt{\max_i f_i}$$

Variance can only increase if influence is non-uniform.
More entertainment

Can add edges to improve social cost (by $\leq \text{PoA}$):

1. Figuring out where to add a little weight is easy
2. Figuring out best edge additions is NP-hard

Changing from quadratic cost is interesting, but harder.