Today: Two applications and an algorithm idea.
Microguitars from Cornell University (1997 and 2003)

- MHz-GHz mechanical resonators
- Impact: smaller, lower-power cell phones
- Also useful for sensors (inertial, chemical)
- ... and really high-pitch guitars
Modeling a Disk Resonator
Modeling a Disk Resonator

\[ V_{in} \]

\[ C_0 \]

\[ L_x \]

\[ C_x \]

\[ R_x \]

\[ V_{out} \]
Modeling a Ringing Disk

- At what frequencies does this vibrate?
- How quickly is the ringing damped?
- What about errors (in numerics or fabrication)?
- How do we answer these questions fast?
Application: Network Tomography

Measure few paths between $n$ hosts in network to infer
- Path properties
- Link properties
- Network topology?
Network Tomography as Linear Algebra

- Example path properties: latency, loss rate, jitter
- Relate path properties $y_{ij}$ to link property $x_l$:

$$ y_{ij} = \sum_{l \in \text{path}} x_l = \sum_l g_{ijl} x_l $$

where $g_{ijl}$ indicates if link $l$ is used on path $i \rightarrow j$.
- Write in matrix form (one path per row): $y = Gx$
Path Dependencies and Matrix Structure

\[(A_i \rightarrow B_i) + (A_1 \rightarrow B_1) = (A_1 \rightarrow B_i) + (A_i \rightarrow B_1)\]
Gravitational potential at mass $j$ from other masses is

$$
\phi_j(x) = \sum_{i \neq j} \frac{Gm_i}{|x_i - x_j|}.
$$

In cluster A, don’t really need everything about B. Just summarize.
A motivating example

Gravitational potential is a linear function of masses

\[
\begin{bmatrix}
\phi_A \\
\phi_B
\end{bmatrix} = \begin{bmatrix}
P_{AA} & P_{AB} \\
P_{BA} & P_{BB}
\end{bmatrix} \begin{bmatrix}
m_A \\
m_B
\end{bmatrix}.
\]

In cluster A, don’t really need everything about B. Just summarize.
That is, represent $P_{AB}$ (and $P_{BA}$) compactly.
Low-rank interactions

Summarize masses in B with a few variables:

\[ z_B = V_B^T m_B, \quad m_B \in \mathbb{R}^{n_B}, z_B \in \mathbb{R}^p. \]

Then contribution to potential in cluster A is \( U_A z_B \). Have

\[ \phi_A \approx P_{AA} m_A + U_A V_B^T m_B. \]

Do the same with potential in cluster B; get system

\[
\begin{bmatrix}
\phi_A \\
\phi_B
\end{bmatrix} =
\begin{bmatrix}
P_{AA} & U_A V_B^T \\
U_B V_A^T & P_{BB}
\end{bmatrix}
\begin{bmatrix}
m_A \\
m_B
\end{bmatrix}.
\]

Idea is the basis of fast \( n \)-body methods (e.g. fast multipole method).
Sparsification

Want to solve $Ax = b$ where $A = S + UV^T$ is sparse plus low rank.

If we knew $x$, we could quickly compute $b$:

$$z = V^T x$$
$$b = Sx + Uz.$$ 

Use the same idea to write $Ax = b$ as a bordered system\(^1\):

$$\begin{bmatrix} S & U \\ V^T & -I \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$ 

Solve this using standard sparse solver package (e.g. UMFPACK).

\(^1\)This is Sherman-Morrison in disguise
Sparsification in gravity example

Suppose we have $\phi$, want to compute $m$ in

$$
\begin{bmatrix}
\phi_A \\
\phi_B \\
0 \\
0
\end{bmatrix} = 
\begin{bmatrix}
P_{AA} & 0 & 0 & U_A \\
0 & P_{BB} & U_B & 0 \\
V_A^T & 0 & -I & 0 \\
0 & V_B^T & 0 & -I
\end{bmatrix}
\begin{bmatrix}
m_A \\
m_B \\
z_A \\
z_B
\end{bmatrix}.
$$

Add auxiliary variables to get

$$
\begin{bmatrix}
\phi_A \\
\phi_B \\
0 \\
0
\end{bmatrix} = 
\begin{bmatrix}
P_{AA} & 0 & 0 & U_A \\
0 & P_{BB} & U_B & 0 \\
V_A^T & 0 & -I & 0 \\
0 & V_B^T & 0 & -I
\end{bmatrix}
\begin{bmatrix}
m_A \\
m_B \\
z_A \\
z_B
\end{bmatrix}.
$$
Parallel sparsification routine (with Tim Mitchell)
  - User identifies low-rank blocks
  - Code factors the blocks and forms a sparse matrix as above

Works pretty well on an example problem (charge on a capacitor)

My goal state: Sparsification of separators for fast PDE solver
Goal state

I want a direct solver for this!
Many cheerful things

- **MEMS-ish**
  - Frequency-response characterization of AFM tip sharpness
  - Opto-mechanical MEMS gyros

- **Network-ish**
  - Fast factorization methods for path matrices
  - Algebraic properties of path matrices
  - Network topology inference from end-to-end measurements

- **Rank-structured matrices**
  - Frameworks for high-performance parallel implementations
  - Robust direct solvers for 3D PDE discretizations
  - Connections to domain decomposition, etc.

- **Eigenstuff**
  - Algorithms and error analysis for resonance problems
  - Provable bounds for 2D soft obstacle scattering poles
  - Resonances for more general open Riemannian manifolds