# Error Bounds and Error Estimates for Nonlinear Eigenvalue Problems

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2 Jun 2008

# Outline

1. One big idea.
2. One little idea.
3. One illustrative example.

# Big picture

$A : \mathbb{C} \to \mathbb{C}^{n \times n}$ analytic in $\Omega$, usually a Laplace or $z$-transform

$$\begin{aligned} \Lambda(A) &:= \{z \in \mathbb{C} : A(z) \text{ singular}\} \\ \Lambda_\epsilon(A) &:= \{z \in \mathbb{C} : \|A(z)^{-1}\| \geq \epsilon^{-1}\} \end{aligned}$$

- $\Lambda(A)$ and $\Lambda_\epsilon(A)$ describe asymptotics, transients of some linear differential or difference equation.
- Lots of function theoretic proofs from analyzing ordinary eigenvalue problems carry over without change.

# Counting eigenvalues

If $A$ nonsingular on $\Gamma$, analytic inside, count eigs inside by

$$\begin{aligned} W_\Gamma(\det(A)) &= \frac{1}{2\pi i}\int_\Gamma \frac{d}{dz} \ln\det(A(z))\,dz \\ &= \operatorname{tr}\left(\frac{1}{2\pi i}\int_\Gamma A(z)^{-1}A'(z)\,dz\right) \end{aligned}$$

Suppose $E$ also analytic inside $\Gamma$. By continuity,

$$W_\Gamma(\det(A)) = W_\Gamma(\det(A+sE))$$

for $s$ in neighborhood of 0 such that $A+sE$ remains nonsingular on $\Gamma$.

# Idea 1

Winding number counts give continuity of eigenvalues $\implies$
Should consider eigenvalues of $A + sE$ for $0 \leq s \leq 1$:

Analyticity of $A$ and $E$ +
Matrix nonsingularity test for $A + sE$ =

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Inclusion region for $\Lambda(A + E)$ +
Eigenvalue counts for connected components of region

# Example: Matrix Rouché

$\|A^{-1}(z)E(z)\| < 1$ on $\Gamma \implies$ same eigenvalue count in $\Gamma$

Proof:
$\|A^{-1}(z)E(z)\| < 1 \implies A(z) + sE(z)$ invertible for $0 \leq s \leq 1$.

(Gohberg and Sigal proved a more general version in 1971.)

# Example: Nonlinear Gershgorin

Define

$$G_i = \left\{ z : |a_{ii}(z)| < \sum_{j \neq i} |a_{ij}(z)| \right\}$$

Then

1. $\Lambda(A) \subset \cup_i G_i$
2. Connected component $\cup_{i=1}^{m} G_i$ contains $m$ eigs (if bounded and disjoint from $\partial\Omega$)

Proof: Write $A = D + F$ where $D = \mathrm{diag}(A)$.
$D + sF$ is diagonally dominant (so invertible) off $\cup_i G_i$.

# Example: Pseudospectral containment

Define $D = \{z : \|E(z)\| < \epsilon\}$. Then

1. $\Lambda(A + E) \subset \Lambda_\epsilon(A) \cup D^C$
2. A bounded component of $\Lambda_\epsilon(A)$ strictly inside $D$ contains the same number of eigs of $A$ and $A + E$.

# Idea 2

Can use the usual proof to get first-order changes to isolated nonlinear eigenvalues. Let $E$ be a function perturbing $A$. If $A(\lambda)v = 0$ and $w^* A(\lambda) = 0$, then

$$\begin{aligned} 0 &= \delta(w^* A(\lambda) v) \\ &= w^* A(\lambda) \delta v + (\delta w)^* A(\lambda) v + w^* \delta(A(\lambda)) v \\ &= w^* \delta(A(\lambda)) v \\ &= w^* (E(\lambda) + A'(\lambda)\, \delta\lambda) v \end{aligned}$$

So nonlinear eigenvalue changes like

$$\delta\lambda = \frac{w^* E(\lambda) v}{w^* A'(\lambda) v}$$

# Example: Lattice Schrödinger

Consider the discrete analogue to Schrödinger's equation:

$$H\psi = (-T + V)\psi = E\psi$$

where

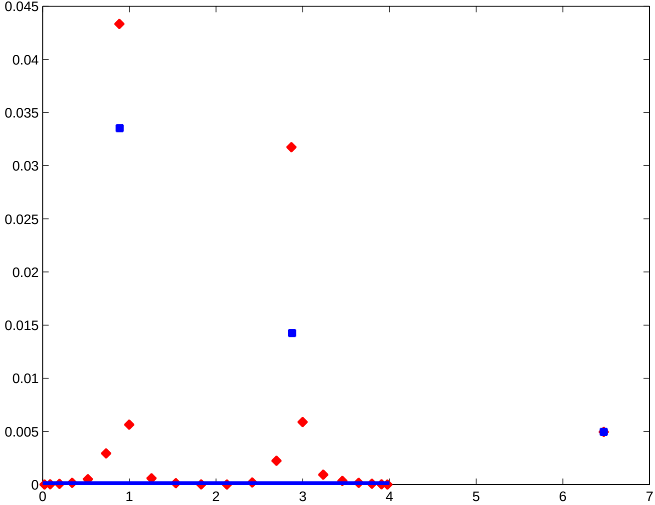
$$(H\psi)_k = -\psi_{k-1} + 2\psi_k - \psi_{k+1} + V_k\psi_k.$$

Assume $V_k = 0$ for $k \leq 0$ and $k \geq L$. May be complex.

Want to relate the spectrum for two variants:

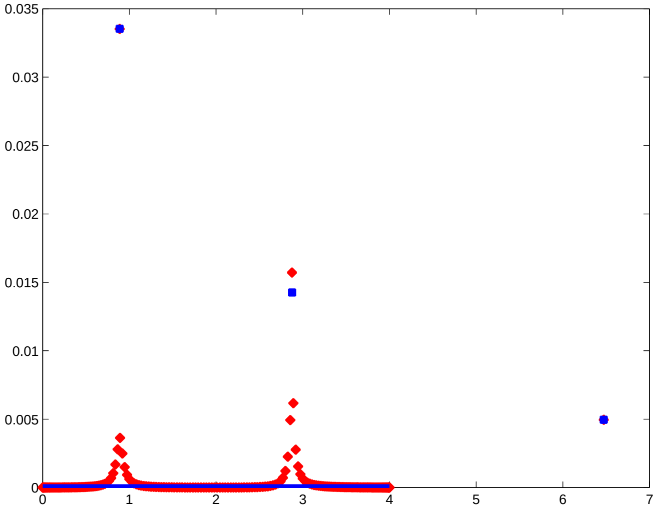
1. Non-negative integers: $\psi_0 = 0$ and $\psi \in l^2$
2. Bounded: $\psi_k = 0$ for $k = 0$ and $k \geq L + N$.

# Example: Lattice Schrödinger




For $V_1 = 0.1i$ and $V_2 = 4$, $N = 20$.

# Example: Lattice Schrödinger




For $V_1 = 0.1i$ and $V_2 = 4$, $N = 200$.

# Spectral Schur complement

Write $H$ in either case as

$$H = \begin{bmatrix} -T_{11} + V_{11} & -e_L e_1^T \\ -e_L e_1^T & -T_{22} \end{bmatrix}$$

Then $\Lambda(H) \cap \Lambda(-T_{22})^c = \Lambda(S)$, where

$$S(z) = (-T_{11} + V_{11}) - zI - \left( e_1^T (-T_{22} - zI)^{-1} e_1 \right) e_L e_L^T$$

Write $S^{(N)}(z)$ and $S^{(\infty)}(z)$ for bounded and unbounded cases.

# Spectral Schur complement

For $z \notin [0,4]$, choose $\xi^2 - (2-z)\xi + 1 = 0$, $|\xi| < 1$. Then

$$\begin{aligned} S^{(\infty)}(z) &= (-T_{11} + V_{11}) - zI - \xi e_L e_L^T \\ S^{(N)}(z) &= (-T_{11} + V_{11}) - zI - \xi \left( \frac{1 - \xi^{2N}}{1 - \xi^{2(N+1)}} \right) e_L e_L^T \end{aligned}$$

Convenient to write $z = 2 - \xi - \xi^{-1}$, use $\xi$ as primary variable.

# Error bounds

Find $\|S^{(\infty)} - S^{(N)}\| \leq \epsilon$ if

$$|\xi| < \left(1 + \frac{\log(3\epsilon^{-1})}{2N+1}\right)^{-1} = 1 - O\left(\frac{\log(\epsilon^{-1})}{N}\right).$$

Therefore, eigenvalues in bounded case (in $\xi$ plane) either

1. Are within $O(\log(\epsilon^{-1})/N)$ of circle (continuous spectrum)
2. Are in $\Lambda_\epsilon(S^{(\infty)})$.

Get exponential convergence to discrete spectrum, linear convergence to continuous spectrum.

# Error estimate

If $S^{(\infty)}$ has an isolated eigenvalue at $\gamma$, then $S^{(N)}$ asymptotically has eigenvalues $\gamma^{(N)} \to \gamma$ with

$$\gamma^{(N)} - \gamma = \gamma^{2N} \frac{w^* e_L e_L^T v_L}{(1-\gamma^2) w^* v - w^* e_L e_L^T v} + O(\gamma^{2N+1})$$

where $S^{(\infty)}(\gamma) v = 0$ and $w^* S^{(\infty)}(\gamma) = 0$.

# Similar applications

- Resonance calculations, error analysis, and some asymptotics for (continuum) 1D Schrödinger problems (joint with M. Zworski)
- Error analysis of resonance calculations via radiation boundary conditions.
- Linear stability analysis for traveling waves.
- Bounds on distance to instability via subspace projections.
- Estimates of damping in MEMS resonators.

# Conclusions

- For analytic NEPs, get analogues to standard perturbation bounds (Rouché, Gerschgorin, pseudospectral)
- Also get first-order perturbation theory
- Get interesting problems via approximation of spectral Schur complements
- Get interesting questions from audience?