Numerical and semi-analytical structure-preserving model reduction for MEMS

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Example Resonant System
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The Dream

Ideally, would like

- Simple models for behavioral simulation
- Parameterized for design optimization
- Including all relevant physics
- With reasonably fast and accurate set-up

We aren’t there yet.

Major theme: use problem structure for better models
Model with radiation damping (PML) gives complex problem:

\[(K - \omega^2 M)u = f, \text{ where } K = K^T, M = M^T\]

Forced solution \(u\) is a stationary point of

\[
I(u) = \frac{1}{2} u^T(K - \omega^2 M)u - u^T f.
\]

Eigenvalues of \((K, M)\) are stationary points of

\[
\rho(u) = \frac{u^T Ku}{u^T Mu}
\]

First-order accurate vectors \(\implies\) second-order accurate eigenvalues.
Disk Resonator Mesh

- Axisymmetric model with bicubic mesh
- About 10K nodal points in converged calculation
Symmetric ROM Accuracy

Results from ROM (solid and dotted lines) near indistinguishable from full model (crosses)
## Symmetric ROM Accuracy

| Frequency (MHz) | $|H(\omega) - H_{\text{reduced}}(\omega)|/|H(\omega)|$ |
|----------------|---------------------------|
| 45             | $10^{-6}$                  |
| 46             | $10^{-4}$                  |
| 47             | $10^{-2}$                  |

Preserve structure $\Rightarrow$ get twice the correct digits

**Arnoldi ROM**

- Structure-preserving ROM
- Arnoldi ROM
Variation in Quality of Resonance

Simulation and lab measurements vs. disk thickness
Dimensionless continuum equations for thermoelastic damping:

\[
\begin{align*}
\sigma &= \hat{C} \epsilon - \xi \theta 1 \\
\ddot{u} &= \nabla \cdot \sigma \\
\dot{\theta} &= \eta \nabla^2 \theta - \text{tr}(\dot{\epsilon})
\end{align*}
\]

Discrete equations:

\[
\begin{align*}
M_{uu} \ddot{u} + K_{uu} u &= \xi K_{u\theta} \theta + f \\
C_{\theta\theta} \dddot{\theta} + \eta K_{\theta\theta} \theta &= -C_{\theta u} \dot{u}
\end{align*}
\]

Dimensionless coupling \(\xi\) and heat diffusivity \(\eta\) are \(10^{-4}\) \(\implies\) perturbation method (about \(\xi = 0\)).
Perturbative Mode Calculation

Discretized mode equation:

\[
(-\omega^2 M_{uu} + K_{uu})u = \xi K_{u\theta}\theta \\
(i\omega C_{\theta\theta} + \eta K_{\theta\theta})\theta = -i\omega C_{\theta u}u
\]

First approximation about \(\xi = 0\):

\[
(-\omega_0^2 M_{uu} + K_{uu})u_0 = 0 \\
(i\omega_0 C_{\theta\theta} + \eta K_{\theta\theta})\theta_0 = -i\omega_0 C_{\theta u}u_0
\]

First-order correction in \(\xi\):

\[
-\delta(\omega^2) M_{uu}u_0 + (-\omega_0^2 M_{uu} + K_{uu})\delta u = \xi K_{u\theta}\theta_0
\]

Multiply by \(u_0^T\):

\[
\delta(\omega^2) = -\xi \left( \frac{u_0^T K_{u\theta}\theta_0}{u_0^T M_{uu}u_0} \right)
\]
Thermoelastic Damping Example
We work with hand-build model reduction all the time!
- Circuit elements: Maxwell equation + field assumptions
- Beam theory: Elasticity + kinematic assumptions
- Axisymmetry: 3D problem + kinematic assumption

Idea: Provide *global shapes*
- User defines shapes through a callback
- Mesh serves defines a quadrature rule
- Reduced equations fit known abstractions
Global Shape Functions

Normally:

\[ u(X) = \sum_j N_j(X) \hat{u}_j \]

Global shape functions:

\[ \hat{u} = \hat{u}^l + G(\hat{u}^g) \]

Then constrain values of some components of \( \hat{u}^l \), \( \hat{u}^g \).
“Hello, World!”

Which mode shape comes from the reduced model (3 dof)?

(Left: 28 MHz; Right: 31 MHz)
Respecting problem structure is a Good Thing!

- ODE structure
- Complex symmetric structure
- Perturbative structure
- Geometric structure

Result:
Better accuracy, faster set-up, better understanding.