Structure Preserving Model Reduction for Damped Resonant MEMS

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USNCCM 07, 23 Jul 2007
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Resonant MEMS

Microguitars from Cornell University (1997 and 2003)

- MHz-GHz mechanical resonators
- Favorite application: radio on chip
- Close second: really high-pitch guitars
Your cell phone has many moving parts!
What if we replace them with integrated MEMS?
Ultimate Success

“Calling Dick Tracy!”
Example Resonant System

\[ V_{in} \rightarrow \text{AC} \rightarrow \text{DC} \rightarrow V_{out} \]
Example Resonant System
Model Reduction: Basic set-up

Linear time-invariant system:

\[ Mu'' + Ku = b\phi(t) \]
\[ y(t) = p^T u \]

Frequency domain:

\[ -\omega^2 M\hat{u} + K\hat{u} = b\hat{\phi}(\omega) \]
\[ \hat{y}(\omega) = p^T\hat{u} \]

Transfer function:

\[ H(\omega) = p^T(-\omega^2M + K)^{-1}b \]
\[ \hat{y}(\omega) = H(\omega)\hat{\phi}(\omega) \]
Model Reduction: Basic set-up

Have a *rational* transfer function relating input and output:

\[ H(\omega) = p^T(-\omega^2 M + K)^{-1} b \]

Can approximate \( H \) by Galerkin projection:

\[ \hat{H}(\omega) = (Vp)^T(-\omega^2 V^T M V + V^T K V)^{-1}(Vb) \]

Could also try to approximate \( H \) directly (often equivalent).
The Designer’s Dream

Ideally, would like
- Compact models for behavioral simulation
- Parameterized for design optimization
- Including all relevant physics
- With reasonably fast and accurate set-up

We aren’t there yet.
Approximate $H$ by Galerkin projection:

\[
\hat{H}(i\omega) = (Vp)^T(-\omega^2 V^T MV + V^T KV)^{-1}(Vb)
\]

1. Define $K_\sigma := K - \sigma^2 M$; build a Krylov subspace

   \[
   \text{span}(V) = \mathcal{K}_n(K_\sigma^{-1} M, K_\sigma^{-1} b) = \text{span}\{(K_\sigma^{-1} M)^j K_\sigma^{-1} b\}_{j=0}^n
   \]

   Has the *moment-matching property*:

   \[
   H^{(k)}(i\sigma) = \hat{H}^{(k)}(i\sigma), \quad k = 0, \ldots, n
   \]

   Get $2n$ moments for symmetric systems (or for separate left and right subspaces).

2. Project onto one or more modal vectors.
The Hero of the Hour

Major theme: use problem structure for better reduced models

- ODE structure
- Complex symmetric structure
- Perturbative structure
- Geometric structure
SOAR and ODE structure

Damped second-order system:

\[ Mu'' + Cu' + Ku = P\phi \]
\[ y = V^T u. \]

Projection basis \( Q_n \) with Second Order ARnoldi (SOAR):

\[ M_nu''_n + C_nu'_n + Ku_n = P_n\phi \]
\[ y = V_n^T u \]

where \( P_n = Q_n^T P, \ V_n = Q_n^T V, \ M_n = Q_n^T M Q_n, \ldots \)
Checkerboard Resonator
Checkerboard Resonator

- Anchored at outside corners
- Excited at northwest corner
- Sensed at southeast corner
- Surfaces move only a few nanometers
Performance of SOAR vs Arnoldi

\[ N = 2154 \rightarrow n = 80 \]

Bode plot

- **Magnitude**
- **Phase (degree)**

Exact
SOAR
Arnoldi
Complex Symmetry

Model with radiation damping (PML) gives complex problem:

\[(K - \omega^2 M)u = f, \text{ where } K = K^T, M = M^T\]

Forced solution \(u\) is a stationary point of

\[I(u) = \frac{1}{2} u^T (K - \omega^2 M)u - u^T f.\]

Eigenvalues of \((K, M)\) are stationary points of

\[\rho(u) = \frac{u^T Ku}{u^T Mu}\]

First-order accurate vectors \(\Rightarrow\) second-order accurate eigenvalues.
Disk Resonator Simulations
Disk Resonator Mesh

- Axisymmetric model with bicubic mesh
- About 10K nodal points in converged calculation
Symmetric ROM Accuracy

Results from ROM (solid and dotted lines) near indistinguishable from full model (crosses)
Symmetric ROM Accuracy

Preserve structure $\implies$ get twice the correct digits

| Frequency (MHz) | $|H(\omega) - H_{\text{reduced}}(\omega)||/H(\omega)|$
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Dimensionless continuum equations for thermoelastic damping:

\[
\begin{align*}
\sigma &= \hat{C}\varepsilon - \xi \theta 1 \\
\ddot{u} &= \nabla \cdot \sigma \\
\dot{\theta} &= \eta \nabla^2 \theta - \text{tr}(\dot{\varepsilon})
\end{align*}
\]

Dimensionless coupling \(\xi\) and heat diffusivity \(\eta\) are \(10^{-4}\) \implies perturbation method (about \(\xi = 0\)).

Large, non-self-adjoint, first-order coupled problem \(\rightarrow\) Smaller, self-adjoint, mechanical eigenproblem + symmetric linear solve.
Thermoelastic Damping Example
Performance for Beam Example

The graph illustrates the comparison between the Perturbation method and the First-order form for different beam lengths. The x-axis represents the beam length in microns, ranging from 10 to 100. The y-axis represents the time in seconds, ranging from 0 to 10.

- The solid line represents the Perturbation method.
- The dashed line represents the First-order form.

The graph shows a linear relationship between beam length and time for both methods.
Aside: Effect of Nondimensionalization

100 \( \mu m \) beam example, first-order form.

Before nondimensionalization
- Time: 180 s
- \( \text{nnz}(L) = 11M \)

After nondimensionalization
- Time: 10 s
- \( \text{nnz}(L) = 380K \)
We work with hand-build model reduction all the time!

- Circuit elements: Maxwell equation + field assumptions
- Beam theory: Elasticity + kinematic assumptions
- Axisymmetry: 3D problem + kinematic assumption

Idea: Provide *global shapes*

- User defines shapes through a callback
- Mesh serves defines a quadrature rule
- Reduced equations fit known abstractions
Global Shape Functions

Normally:

\[ u(X) = \sum_j N_j(X) \hat{u}_j \]

Global shape functions:

\[ \hat{u} = \hat{u}^l + G(\hat{u}^g) \]

Then constrain values of some components of \( \hat{u}^l, \hat{u}^g \).
“Hello, World!”

Which mode shape comes from the reduced model (3 dof)?

(Left: 28 MHz; Right: 31 MHz)
Respecting problem structure is a Good Thing!

- ODE structure
- Complex symmetric structure
- Perturbative structure
- Geometric structure

Result:
Better accuracy, faster set-up, better understanding.