# Continuation of Sparse Eigendecompositions

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# Basic setting

Have a $C^k$ function

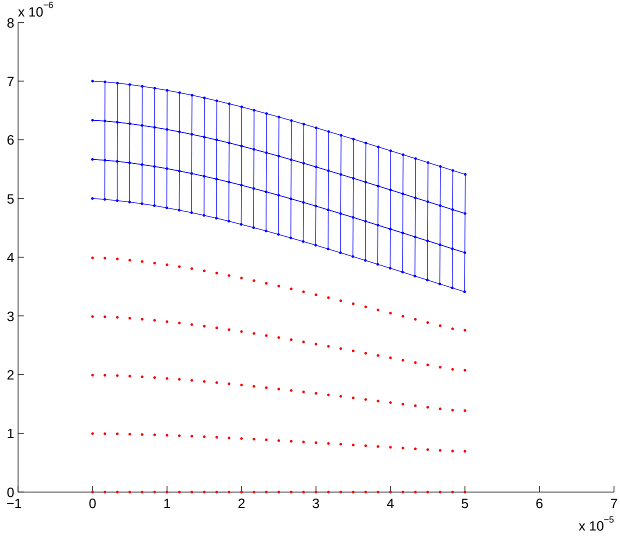
$$A : [0, 1] \to \mathbb{R}^{n \times n}$$

Want to compute eigenvectors $v(s)$ and values $\lambda(s)$ for $A(s)$. More generally, want an invariant subspace basis $V(s)$.

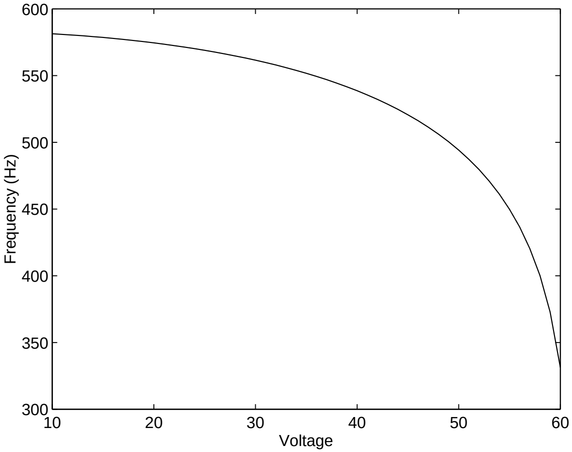
Applications:

1. Resonant system design
2. Bifurcation analysis

# Example: Cantilever tuning



# Example: Cantilever tuning



# Example: Belousov-Zhabotinski reaction

www.pojman.com/NLCD-movies/NLCD-movies.html

# Reaction-diffusion models

$$\frac{\partial u}{\partial t} = D\nabla^2 u + F(u; s)$$

Describes many systems:

- Chemical reactions (like the B-Z reaction)
- Signals in nerves
- Ecological systems
- Phase transitions

See *Chemical Oscillations, Waves, and Turbulence* (Kuramoto).

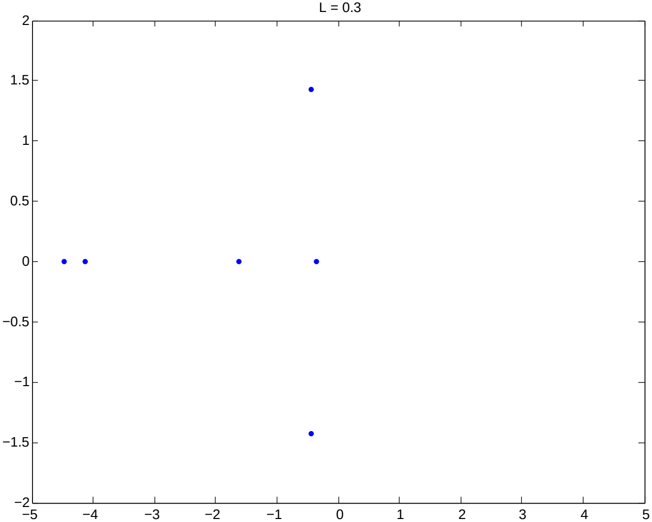
# Stability analysis

Linearize about an equilibrium branch $u_0(s)$:

$$\frac{\partial}{\partial t}\delta u = \left(D\nabla^2 + F_u(u_0(s); s)\right)\delta u = A(s)\,\delta u$$

- Stable if eigenvalues of $A(s)$ have negative real part
- When stability changes, have a *bifurcation*
- Complex eigs cross imaginary axis $\implies$ oscillations, a *Hopf bifurcation*

# Hopf bifurcation in the Brusselator



# Subspaces and stability analysis

- Diagnose stability from a small subspace (slow dynamics)
- Idea: Continue invariant subspace along with the solution
- Problem: Switching subspaces
- Problem: Missing information

# CIS algorithm

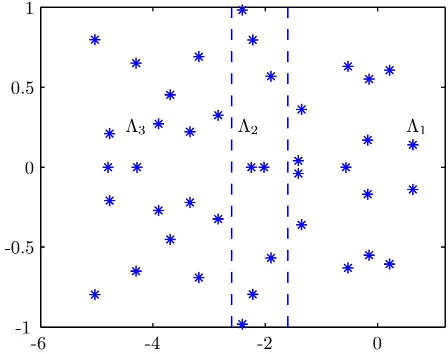
Compute a continuous block Schur form

$$Q(s)^T A(s) Q(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix}$$

Algorithm phases:

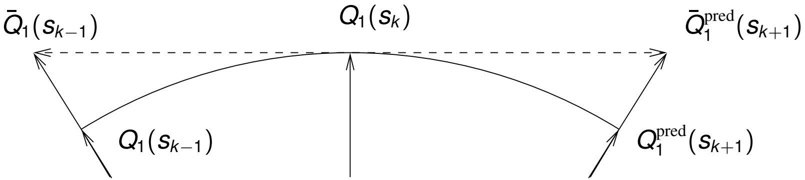
- Initialize
- Predict
- Correct
- Normalize
- Adapt

# Initialization




- Compute rightmost part of the spectrum
- Include all unstable eigenvalues + a few stable ones
- Keep eigenvalue clusters together
  (prevent artificially short steps)

# Prediction




- Normalize to tangent plane:

$$\bar{Q}_1(s) = Q_1(s)\left(Q_1(s_k)^T Q_1(s)\right)^{-1}$$

- Predict $\bar{Q}_1(s_{k+1})$ by polynomial fitting through $\bar{Q}_1(s_k), \bar{Q}_1(s_{k-1}), \ldots$.
- Suggests projection space should include computed spaces from previous few steps.

# Correction

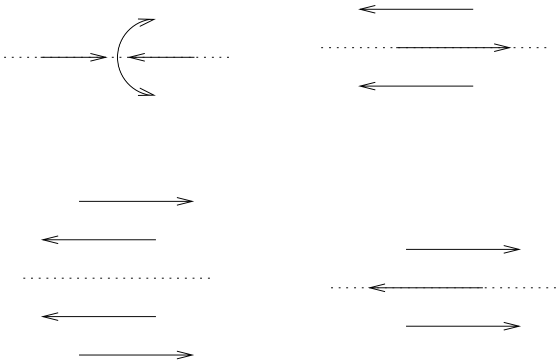
Solve the nonlinear equations

$$\begin{aligned} AQ_1 - Q_1 T_{11} &= 0 \\ (Q_1^{\text{prev}})^T Q_1 - I &= 0 \end{aligned}$$

- Linearization is a bordered Sylvester equation
- Newton $\approx$ block RQ iteration
- Modified Newton $\approx$ subspace iteration
- Or extract from a Krylov subspace

Then normalize to minimize Frobenius change in $Q_1(s)$.

# Adaptation



May need to adjust space if

- Real parts of continued eigenvalues overlap the rest of the spectrum (generic possibilities shown)
- Eigenvalues cross imaginary axis (bifurcation)

# Testing continuity

How to ensure proposed basis spans the right subspace?

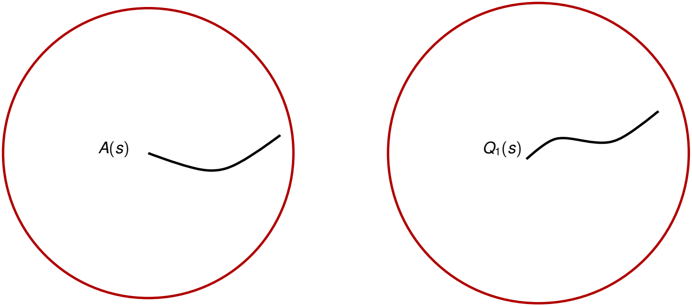
- Check rate of Newton convergence
- Check angles between subspaces
- Check distance between eigenvalues

Anything less heuristic?

# Perturbation approaches

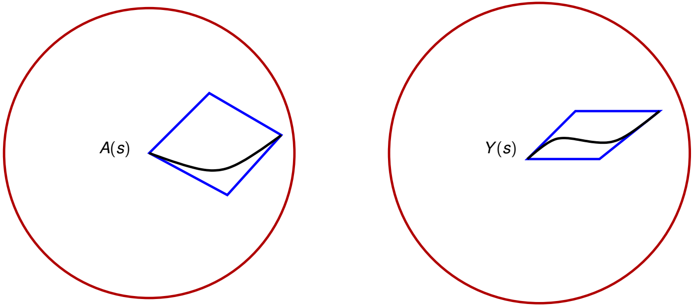

$A(s)$ and $Q_1(s)$ trace some paths in matrix spaces.

# Perturbation approaches




Can determine that if $A(s)$ stays in some region, $Q_1(s)$ is well defined and stays in some other region.

# Perturbation approaches




Can we test with smaller regions?

# Checking the subspace

Recall:

$$\mathrm{sep}(B, C) = \|\mathrm{S}^{-1}\|^{-1}, \text{ where } \mathrm{S}(X) = BX - XC$$

If for $s \in [0, h]$,

$$\mathrm{sep}(A_{11}, A_{22})^2 > 4\|A_{12}\|\|A_{21}\|$$

Then have a unique $C^k$ invariant subspace associated with $A_{11}$.

So if $A_{21}(0) = 0$ then $Q_1(0) = [I; 0]$ extends to $Q_1(s)$

# Checking the subspace

Have block Schur $Q^T A(s) Q = T$ at $s_k$ and $s_{k+1} = s_k + h$.

- Geodesic interpolation $U(s)$ between $Q(s_k)$ and $Q(s_{k+1})$.
- Similarity: $\hat{T}(s) = U(s)^T A(s) U(s)$.

If $\theta_{\max}$ = largest angle between $Q(s_k)$ and $Q(s_{k+1})$,

$$\|\dot{\hat{T}}\| \leq 2\theta_{\max}\|A\| + \|\dot{A}\|.$$

Then get interpolation bounds:

- $\operatorname{sep}(\hat{T}_{11}, \hat{T}_{22}) = O(1)$
- $\hat{T}_{12} = O(1)$
- $\hat{T}_{21} = O(h^2)$

# Checking the subspace

Check that for all $s \in [0, h]$,

$$\text{sep}(\hat{T}_{11}, \hat{T}_{22})^2 > 4\|\hat{T}_{12}\|\|\hat{T}_{21}\|$$

So test based on:

- Conditioning of subspace (spectral separation)
- Measure of non-normality ($\|\hat{T}_{12}\|$)
- Residual from interpolating ($\|\hat{T}_{21}\|$)

# Are we there yet?

Believe we can compute

$$Q(s)^T A(s) Q(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix}$$

What if the space isn't rich enough ($T_{22}$ unstable)?

Want conditions for $T_{22}$ stable.

- *Do not* want another nonsymmetric eigenproblem
- Only need sufficient conditions

Use the fact that we expect rapid decay in most modes.

# Stability of $T_{22}$

Recall: if $\dot{u} = T_{22}u$ then

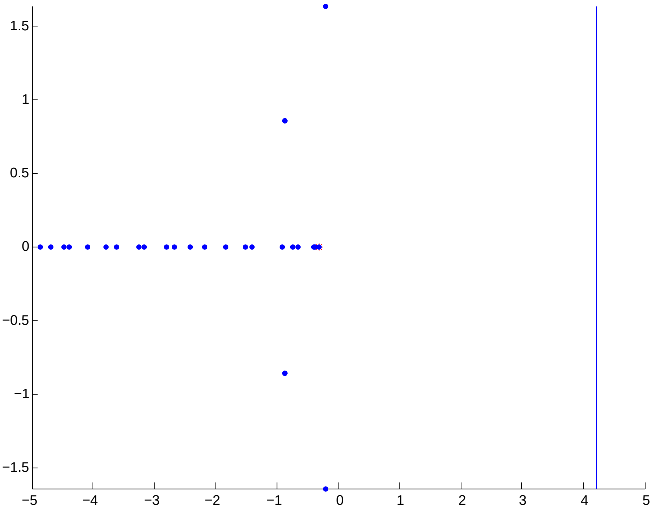
$$\frac{d}{dt}\|u\|^2 = 2u^T T_{22} u.$$

Define spectral abscissa

$$\omega(T_{22}) = \max_v \{v^T T_{22} v\} = \lambda_{\max}(H(T_{22}))$$

Finding $\omega(T_{22})$ is a symmetric exterior eigenvalue problem!
$\implies$ estimate with Lanczos.

# Bound applied to a 2D Brusselator



# Conclusions

- Continuing eigendecompositions is useful for resonator design and for bifurcation analysis
- Basic algorithm: predictor-corrector + Krylov subspaces
- Tests to make sure computed subspace is good enough
- Ongoing software work (CL-MATCONT extension)