Eigenvalues, Resonance Poles, and Damping in MEMS

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Outline

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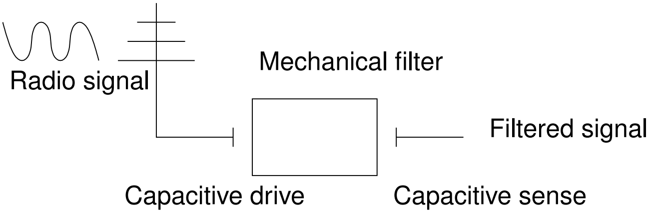
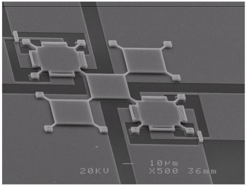
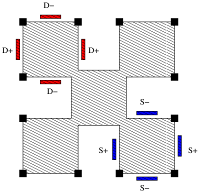
Resonating MEMS

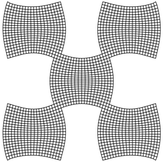


- Mechanical high-frequency (high MHz-GHz) filter
  - Your cell phone is mechanical!
  - New MEMS filters can be integrated with circuitry
    $\Longrightarrow$ smaller and lower power
- Can also make MEMS frequency references, resonant sensors, ...

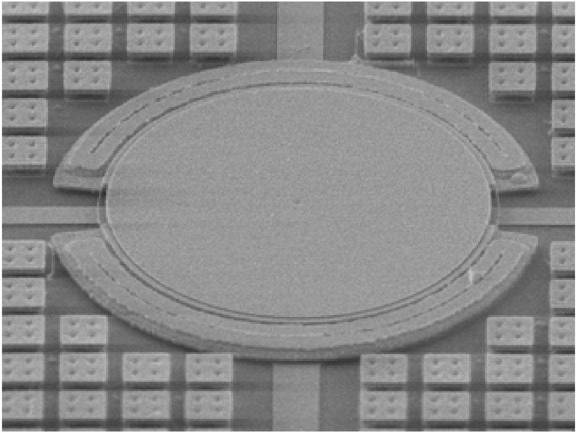
Resonant MEMS: Checkerboard





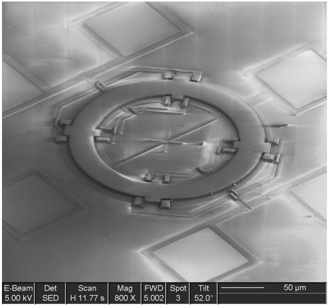


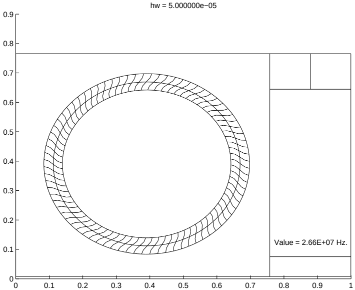
Resonant MEMS: Disk resonator



SiGe disk resonators built by E. Quévy

Resonant MEMS: Shear ring





Transfer Functions

Time domain:

$$\begin{aligned} Mu'' + Bu' + Ku &= b\phi(t) \\ y(t) &= p^T u \end{aligned}$$

Laplace domain:

$$\begin{aligned} s^2 M\hat{u} + sB\hat{u} + K\hat{u} &= b\hat{\phi}(s) \\ \hat{y}(s) &= p^T \hat{u} \end{aligned}$$

Transfer function:

$$\begin{aligned} H(s) &= p^T (s^2 M + sB + K)^{-1} b \\ \hat{y}(s) &= H(s)\hat{\phi}(s) \end{aligned}$$

Think Globally, Approximate Locally

Want to treat subsystems like simple circuits

- Pick a simple circuit topology
- Pick a target resonant frequency ω_0
- Approximate $H(s)$ near $i\omega_0$ by circuit transfer function

Example: Equivalent Circuit for Disk

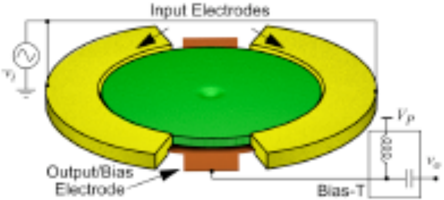
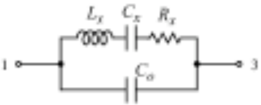


Fig. 8. Perspective view of a one-port measurement setup. (Color version available online at http://ieeexplore.ieee.org.)



$R_x = r_x/(4\eta_1^2)$

$L_x = l_x/(4\eta_1^2)$

$C_x = (4\eta_1^2)c_x$

Fig. 9. Equivalent circuit for the radial-contour mode disk operated as a one-port.

(Clark, Hsu, Abdelmoneum, Nguyen. JMEMS 11 (6))

Damping and Q

Designers want high *quality of resonance* (Q)

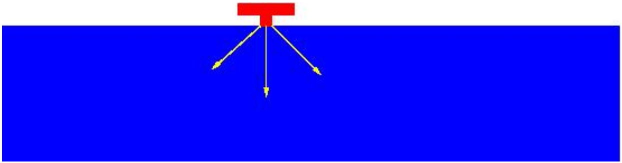
- Dimensionless damping in a one-dof system

$$\frac{d^2u}{dt^2} + Q^{-1}\frac{du}{dt} + u = F(t)$$

- For a resonant mode with frequency $\omega \in \mathbb{C}$:

$$Q := \frac{|\omega|}{2\,\mathrm{Im}(\omega)} = \frac{\text{Stored energy}}{\text{Energy loss per radian}}$$

Damping Mechanisms



Possible loss mechanisms:

- Fluid damping
- Material losses
- Thermoelastic damping
- Anchor loss

Outline

... Where Angels Fear to Tread

Wait a moment:

- When is a domain “effectively semi-infinite?”
- What do the model modes mean?
- How do I estimate the overall accuracy?

Simple String Model

One-dimensional wave problem:

$$\begin{aligned} s^2 \rho u &= \sigma_x, \; x \in (0,1) \\ \sigma &= E u_x \\ u(0,s) &= u_0(s) \\ u(1,s) &= 0. \end{aligned}$$

Solutions have the form

$$\begin{aligned} u(x,s) &= d \sin k(x-1) \\ d &= -1/\sin(k) \\ s &= i\omega \\ k &= \omega\sqrt{\rho/E} = \omega/c. \end{aligned}$$

Dynamic Stiffness

Dynamic stiffness at $x = 0$ is

$$\begin{aligned} \kappa(s) &:= \frac{\sigma(0,s)}{u(0,s)} \\ &= -Ek\cot(k) \end{aligned}$$

where $k = \omega/c = \omega\sqrt{\rho/E}$

Poles at $k = n\pi$, $n \neq 0$.

Viscoelastic String Model

Now make the constitutive relation hysteretic:

$$\begin{aligned} s^2 \rho u &= \sigma_x, \quad x \in (0,1) \\ \sigma &= E(s) u_x \\ u(0,s) &= u_0(s) \\ u(1,s) &= 0. \end{aligned}$$

where $E(s)$ has poles only on the negative real s axis.

Free vibrations solve a *nonlinear* eigenproblem.

Dynamic Stiffness

As before, dynamic stiffness is

$$\begin{aligned} \kappa(s) &:= \frac{\sigma(0,s)}{u(0,s)} \\ &= -E(s)k\cot(k) \end{aligned}$$

but now $k = \omega/c = \omega\sqrt{\rho/E(i\omega)}$.

Perturbation for Isolated Singularities

Where $E(i\omega) \approx E_0$, E_0 a real constant, can approximate (isolated) poles by perturbation. Poles at

$$\omega \approx \omega_0 \left(1 + 2 \tan \delta\right)$$

where

$$\begin{aligned} \omega_0 &= n\pi c_0 = n\pi \sqrt{E_0/\rho} \\ \tan \delta &= \frac{\text{Im}(E(i\omega_0))}{\text{Re}(E(i\omega_0))}. \end{aligned}$$

Outline

Long Viscoelastic String

Now change length:

$$\begin{aligned} s^2 \rho u &= \sigma_x, \; x \in (0, L) \\ \sigma &= E(s) u_x \\ u(0, s) &= u_0(s) \\ u(L, s) &= 0. \end{aligned}$$

Dynamic stiffness becomes

$$\kappa_L = -Ek \cot(Lk)$$

Dynamic Stiffness as $L \to \infty$

Consider asymptotic behavior:

$$
\begin{aligned}
\kappa_L &= -Ek\cot(Lk) \\
&= -ikE\frac{e^{ikL}+e^{-ikL}}{e^{ikL}-e^{-ikL}} \\
&= \begin{cases} -ikE(1+O(e^{-2ikL})) \\ -ikE(-1+O(e^{2ikL})) \end{cases}
\end{aligned}
$$

Pointwise convergence almost everywhere to

$$
\kappa_\infty = \begin{cases} -ikE, & \mathrm{Im}(k) > 0 \\ ikE, & \mathrm{Im}(k) < 0 \end{cases}
$$

No convergence along $\mathrm{Im}(k) = 0$.

Dynamic Stiffness as $L \to \infty$

Limiting function is

$$\kappa_\infty = -ik^+ E$$

where k^+ is the root of

$$k^2 = \omega^2 \frac{E(i\omega)}{\rho}$$

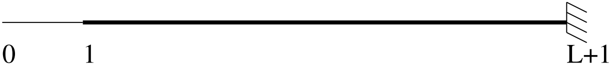
chosen so that $\text{Im}(k^+) > 0$. Have a branch cut along

$$\omega^2 E(i\omega) > 0.$$

From Long to Infinite

- As $L \to \infty$, eigenvalues cluster along a curve
- Away from this curve, κ_L converges pointwise to κ_∞
- Curve is a branch cut in definition of κ_∞

Large Absorbing Domain



$$\begin{aligned}
s^2\rho_1 u(x,s) &= \sigma(x,s),\ x \in (0,1)\\
s^2\rho_2 u(x,s) &= \sigma(x,s),\ x \in (1,L+1)\\
\sigma(x,s) &= E_1(s)u_x(s),\ x \in (0,1)\\
\sigma(x,s) &= E_2(s)u_x(s),\ x \in (1,L+1)\\
u(0,s) &= u_0(s)\\
u(0,L+1) &= 0
\end{aligned}$$

And continuity of u and σ across $x = 1$.

Dynamic Stiffness

$$\kappa_L = -E_1 k_1 \frac{\cos(k_1) + i\tilde{\xi}\sin(k_1)}{\sin(k_1) - i\tilde{\xi}\cos(k_1)}.$$

where

$$\begin{aligned} \delta &= e^{-2ik_2 L} \\ \xi &= \frac{E_1 k_1}{E_2 k_2} = \sqrt{\frac{\rho_1 E_1}{\rho_2 E_2}} \\ \tilde{\xi} &= \xi \frac{1-\delta}{1+\delta}. \end{aligned}$$

If $\xi = 0$, recover the clamped case.

Asymptotic Behavior

Choose k_2^- such that $\mathrm{Im}(k_2^-) < 0$. As $L \to \infty$, $|\delta| \to 0$, and

$$\kappa_\infty = -E_1 k_1 \frac{\cos(k_1) + i\xi \sin(k_1)}{\sin(k_1) - i\xi \cos(k_1)}.$$

Poles of κ_∞ satisfy:

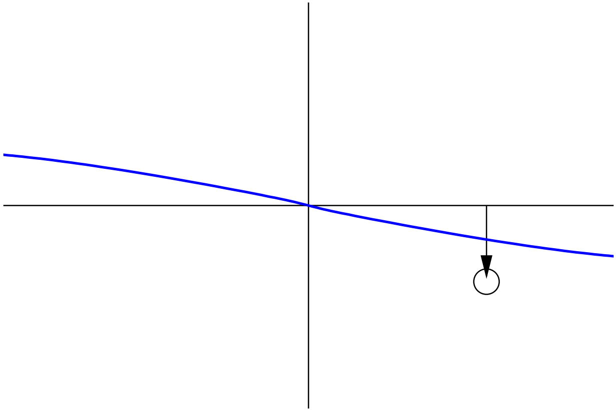
$$\tan(k_1) = i\xi$$

Taylor expand to approximate pole location:

$$k_1 \approx n\pi + i\xi_0.$$

where ξ_0 is the value of ξ at $k_1 = n\pi$.

Resonance Poles



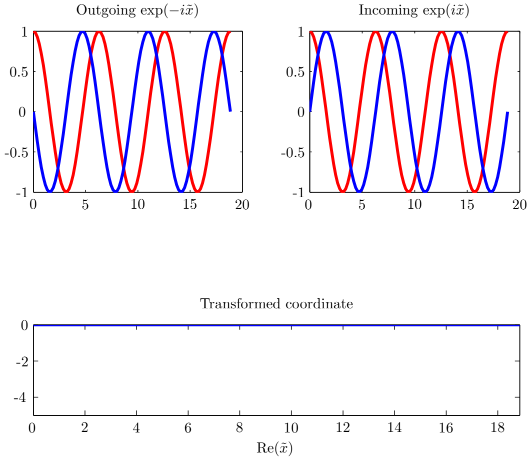
We chose a principle value for k_2 – but the singularity occurs on the other sheet!

Outline

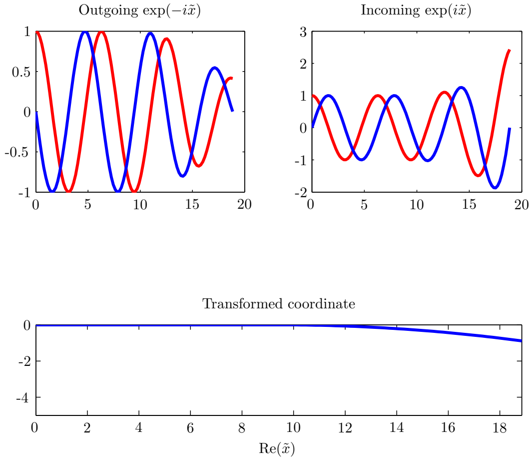
Approximation Steps

- Isolated poles near real $\omega \implies$ simple approximation
- As $L \to \infty$, poles cluster
- As $L \to \infty$, convergence to infinite domain a.e.
- Infinite domain has branch cuts and resonance poles
- Isolated resonance poles near real $\omega \implies$ simple approximation

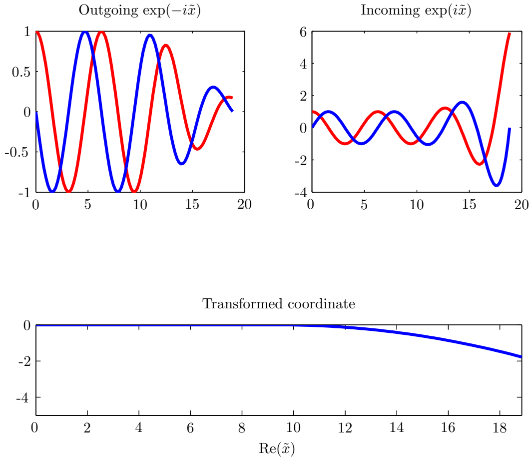
Infinite Domain via PML



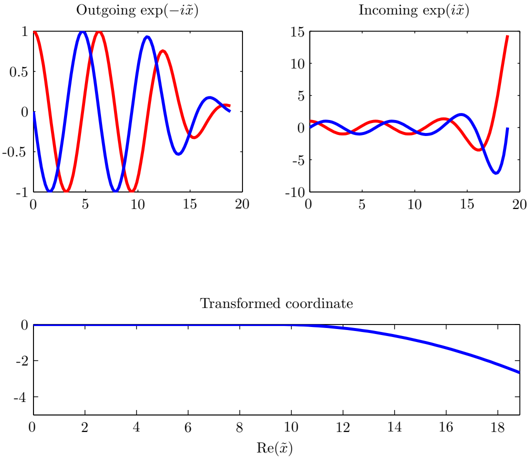
Infinite Domain via PML



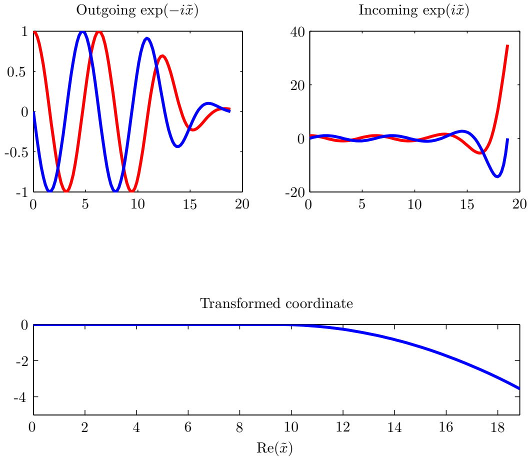
Infinite Domain via PML



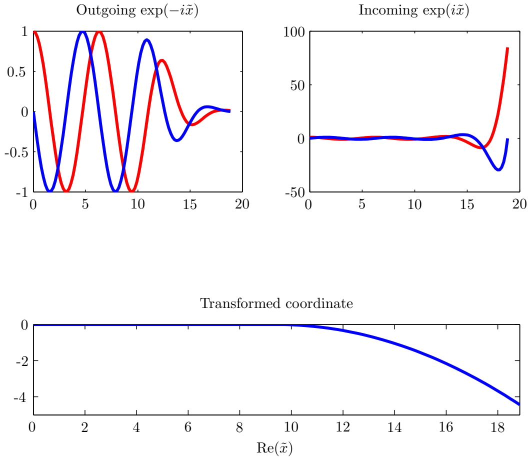
Infinite Domain via PML



Infinite Domain via PML

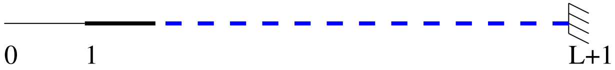


Infinite Domain via PML



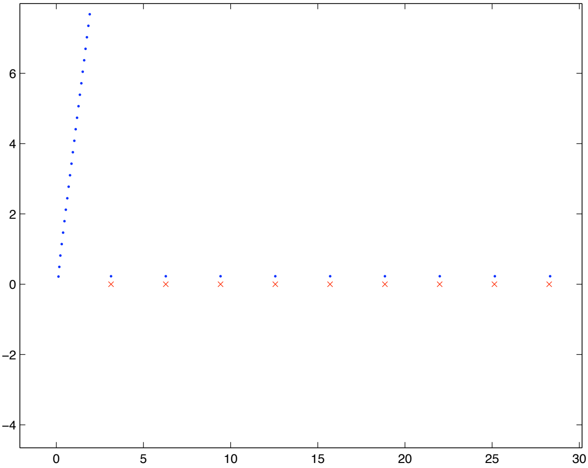
PML and Resonance Poles

Take inhomogeneous problem from before and add PML:



- For real ω, same limiting dynamic stiffness
- But the branch cut is in a different place!
- Means resonance poles become true poles

PML and Resonance Poles



Conclusions

- Modal analysis gives good approximations near isolated poles
- Large, damped domains $\implies$ clustered poles
- *Away* from poles, converges to unbounded problem
- Unbounded problem has branch cuts, resonance poles
- PML moves branch cuts, reveals the resonance poles
- Analysis gives good approximations to original problem near isolated resonance poles