Computer-Aided Design for Micro-Electro-Mechanical Systems
Eigenvalues, Energy Losses, and Dick Tracy Watches

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Caltech, 1 Feb 2006
Application modeling
- Checkerboard resonator
- Disk resonator
- Shear ring resonator

Mathematical analysis
- Physical modeling and finite element technology
- Structured eigenproblems and reduced-order models
- Parameter-dependent eigenproblems

Software engineering
- HiQLab
- SUGAR
- FEAPMEX
The Computational Science Picture

- Application modeling
  - Checkerboard resonator
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- Software engineering
  - HiQLab
  - SUGAR
  - FEAPMEX
What are MEMS?
MEMS Basics

- Micro-Electro-Mechanical Systems
  - Chemical, fluid, thermal, optical (MECFTOMS?)

- Applications:
  - Sensors (inertial, chemical, pressure)
  - Ink jet printers, biolab chips
  - Radio devices: cell phones, inventory tags, pico radio

- Use integrated circuit (IC) fabrication technology

- Tiny, but still classical physics
Microguitars from Cornell University (1997 and 2003)

- MHz-GHz mechanical resonators
- Impact: smaller, lower-power cell phones
- Other uses:
  - Sensing elements (e.g. chemical sensors)
  - Really high-pitch guitars
Micromechanical Filters

- Mechanical high-frequency (high MHz-GHz) filter
  - Your cell phone is mechanical!
  - New MEMS filters can be integrated with circuitry
    \[\Rightarrow\] smaller and lower power
Ultimate Success

“Calling Dick Tracy!”

I’m On My Way
Designing Transfer Functions

Time domain:

\[ Mu'' + Cu' + Ku = b\phi(t) \]
\[ y(t) = p^T u \]

Frequency domain:

\[ -\omega^2 M\hat{u} + i\omega C\hat{u} + K\hat{u} = b\hat{\phi}(\omega) \]
\[ \hat{y}(\omega) = p^T \hat{u} \]

Transfer function:

\[ H(\omega) = p^T (-\omega^2 M + i\omega C + K)^{-1} b \]
\[ \hat{y}(\omega) = H(\omega)\hat{\phi}(\omega) \]
Want “sharp” poles for narrowband filters

$20 \log_{10} |H(\omega)|$

$\omega$

Want to minimize damping
Checkerboard Resonator

- Anchored at outside corners
- Excited at **northwest** corner
- Sensed at **southeast** corner
- Surfaces move only a few nanometers
Checkerboard Model Reduction

- Finite element model: $N = 2154$
  - Expensive to solve for every $H(\omega)$ evaluation!
- Build a reduced-order model to approximate behavior
  - Reduced system of 80 to 100 vectors
  - Evaluate $H(\omega)$ in milliseconds instead of seconds
  - Without damping: standard Arnoldi projection
  - With damping: Second-Order ARnoldi (SOAR)
Checkerboard Simulation

- Frequency (Hz)
- Amplitude (dB)
- Phase (rad)

Graphs showing the simulation results with frequency on the x-axis and amplitude or phase on the y-axis.
Checkerboard Measurement

S. Bhave, MEMS 05
Designers want high quality of resonance ($Q$)

- Dimensionless damping in a one-dof system

$$\frac{d^2 u}{dt^2} + Q^{-1} \frac{du}{dt} + u = F(t)$$

- For a resonant mode with frequency $\omega \in \mathbb{C}$:

$$Q := \frac{|\omega|}{2 \text{Im}(\omega)} = \frac{\text{Stored energy}}{\text{Energy loss per radian}}$$
Enter HiQLab

- Existing codes do not compute quality factors
- ... and awkward to prototype new solvers
- ... and awkward to programmatically define meshes
- So I wrote a new finite element code: HiQLab
HiQLab Structure

- User interfaces (MATLAB, Lua)
- Problem description (Lua)
- Core libraries (C++)
- Solver library (C, C++, Fortran, MATLAB)
- Element library (C++)

- Full scripting language for mesh input
- Callbacks for boundary conditions, material properties
- MATLAB interface for quick algorithm prototyping
- Cross-language bindings are automatically generated
Contributions

- Built predictive model used to design checkerboard
- Used model reduction to get thousand-fold speedup – fast enough for interactive use
- Wrote FEAPMEX to script parameter studies
- Wrote a new code, HiQLab, to study damping
SiGe disk resonators built by E. Quévy
Possible loss mechanisms:
- Fluid damping
- Material losses
- Thermoelastic damping
- Anchor loss

Model substrate as semi-infinite with a Perfectly Matched Layer (PML).
Perfectly Matched Layers

- Complex coordinate transformation
- Generates a “perfectly matched” absorbing layer
- Idea works with general linear wave equations
  - Electromagnetics (Berengér, 1994)
  - Quantum mechanics – *exterior complex scaling* (Simon, 1979)
  - Elasticity in standard finite element framework (Basu and Chopra, 2003)
Domain: \( x \in [0, \infty) \)

Governing eq:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0
\]

Fourier transform:

\[
\frac{d^2 \hat{u}}{dx^2} + k^2 \hat{u} = 0
\]

Solution:

\[
\hat{u} = c_{\text{out}} e^{-ikx} + c_{\text{in}} e^{ikx}
\]
Model with Perfectly Matched Layer Layer

\[
\frac{d\tilde{x}}{dx} = \lambda(x) \text{ where } \lambda(s) = 1 - i\sigma(s)
\]

\[
\frac{d^2\hat{u}}{d\tilde{x}^2} + k^2 \hat{u} = 0
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\[
\hat{u} = c_{\text{out}} e^{-ik\tilde{x}} + c_{\text{in}} e^{ik\tilde{x}}
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Model with Perfectly Matched Layer Layer

\[ \frac{d\tilde{x}}{dx} = \lambda(x) \text{ where } \lambda(s) = 1 - i\sigma(s), \]

\[ \frac{1}{\lambda} \frac{d}{dx} \left( \frac{1}{\lambda} \frac{d\hat{u}}{dx} \right) + k^2 \hat{u} = 0 \]

\[ \hat{u} = c_{\text{out}} e^{-ikx-k\Sigma(x)} + c_{\text{in}} e^{ikx+k\Sigma(x)} \]

\[ \Sigma(x) = \int_0^x \sigma(s) \, ds \]
If solution clamped at $x = L$ then

$$\frac{c_{in}}{c_{out}} = O(e^{-k\gamma}) \text{ where } \gamma = \Sigma(L) = \int_{0}^{L} \sigma(s) \, ds$$
Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$  
Incoming $\exp(i\tilde{x})$

Transformed coordinate

Re($\tilde{x}$)

0 2 4 6 8 10 12 14 16 18

-4 -2 0 2 4 6 8 10 12 14 16 18
Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate
Model Problem Illustrated

Outgoing \( \exp(-i\tilde{x}) \)

Incoming \( \exp(i\tilde{x}) \)

Transformed coordinate

Re(\(\tilde{x}\))
Model Problem Illustrated

Outgoing exp(−i\tilde{x})

Incoming exp(i\tilde{x})

Transformed coordinate

\text{Re}(\tilde{x})

Outgoing exp(−i\tilde{x})

Incoming exp(i\tilde{x})

Transformed coordinate

Re(\tilde{x})
Model Problem Illustrated

Outgoing exp(\(-i\tilde{x}\))

Incoming exp(\(i\tilde{x}\))

Transformed coordinate

\begin{align*}
\text{Outgoing exp(} -i\tilde{x}) & \quad \text{Incoming exp(} i\tilde{x}) \\
\text{Transformed coordinate} &
\end{align*}
Model Problem Illustrated

Outgoing $\exp(-i\tilde{x})$

Incoming $\exp(i\tilde{x})$

Transformed coordinate

\[
\begin{align*}
\text{Re}(\tilde{x}) & \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \\
0 & \quad 5 \quad 10 \quad 15 \quad 20 & -5 \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 & -50 \quad 0 \quad 50 \quad 100 & -1 \quad -0.5 \quad 0 \quad 0.5 \quad 1
\end{align*}
\]
Finite Element Implementation

- Combine PML and isoparametric mappings

\[
\mathbf{k}^e = \int_{\Omega^e} \tilde{\mathbf{B}}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{J} \, d\Omega^e
\]

\[
\mathbf{m}^e = \int_{\Omega^e} \rho \mathbf{N}^T \mathbf{N} \mathbf{J} \, d\Omega^e
\]

- Matrices are complex symmetric
Outline

1. Resonant MEMS
2. Anchor Losses
3. Complex Symmetry
4. Disk Resonator Analysis
Want to know about the transfer function $H(\omega)$:

$$H(\omega) = p^T(K - \omega^2 M)^{-1}b$$

Can either

- Locate poles of $H$ (eigenvalues of $(K, M)$)
- Plot $H$ in a frequency range (Bode plot)

Usual tactic: subspace projection

- Build an Arnoldi basis $V$
- Compute with much smaller $V^*KV$ and $V^*MV$

Can we do better?
Variational Principles

- Variational form for complex symmetric eigenproblems:
  - Hermitian (Rayleigh quotient):
    \[ \rho(v) = \frac{v^* K v}{v^* M v} \]
  - Complex symmetric (modified Rayleigh quotient):
    \[ \theta(v) = \frac{v^T K v}{v^T M v} \]
    
    First-order accurate eigenvectors \( \Rightarrow \)
    Second-order accurate eigenvalues.
    
    Key: relation between left and right eigenvectors.
Accurate Model Reduction

- Build new projection basis from $V$:
  $$W = \text{orth}[\text{Re}(V), \text{Im}(V)]$$

- $\text{span}(W)$ contains both $\mathcal{K}_n$ and $\bar{\mathcal{K}}_n$
  $$\implies$$ double digits correct vs. projection with $V$

- $W$ is a real-valued basis
  $$\implies$$ projected system is complex symmetric
Contributions

• New formulation of perfectly matched layers
  • Easy to apply PML to axisymmetric, 2D, or 3D models
  • Same formulation applies to electromagnetics, etc.
• Analysis of discretization error for perfectly matched layers
  • Results in cheap, automatic parameter optimization
• Structure-preserving model reduction for complex symmetric systems
  • Double accuracy for same work as standard method
Outline

1. Resonant MEMS
2. Anchor Losses
3. Complex Symmetry
4. Disk Resonator Analysis
Disk Resonator Simulations
Axisymmetric model with bicubic mesh
About 10K nodal points in converged calculation
Mesh Convergence

Cubic elements converge with reasonable mesh density
Results from ROM (solid and dotted lines) near indistinguishable from full model (crosses)
Model Reduction Accuracy

\[ |H(\omega) - H_{\text{reduced}}(\omega)| / |H(\omega)| \]

Frequency (MHz)

Arnoldi ROM
Structure-preserving ROM

Preserve structure \( \Rightarrow \) get twice the correct digits
Response of the Disk Resonator
Time-Averaged Energy Flux
Variation in Quality of Resonance

Simulation and lab measurements vs. disk thickness
Interaction of two nearby eigenmodes
Contributions

- Built disk model in HiQLab and verified against lab measurements
- Demonstrated dominance of anchor loss for this device
- Predicted geometric sensitivity of quality factor $Q$ (which was subsequently verified in the lab)
- Explained $Q$ sensitivity in terms of mode interference
Contributions Summary (1)

- Application modeling
  - Finite element models of several devices
  - Discovery of effects of mode interference
  - Importance of anchor loss vs thermoelastic damping

- Mathematical analysis
  - Reformulation of perfectly-matched layers
  - Analysis of discretization and parameter choice in PMLs
  - Complex symmetry-preserving model reduction
  - Perturbation solution for thermoelastic damping

- Software: HiQLab, FEAPMEX, SUGAR
Contributions Summary (2)

- **HiQLab** (about 33000 lines of code)
  - Collaborations at Berkeley, Cornell, Stanford, Bosch

- **FEAPMEX** (about 5000 lines of code)
  - 2400+ page views
  - Used for instrument models, stochastic structural analysis, ultrasonic nondestructive evaluation problems, material parameter identification problems

- **SUGAR** (about 18000 lines of code)
  - 2000+ downloads
  - Used in classes at Berkeley, Cornell, Johns Hopkins
  - Continued research use for design optimization
Other Contributions

- Lowered complexity of roots from $O(n^3)$ to $O(n^2)$
  - Code to go into next LAPACK release
- Developed first sparse subspace continuation code
  - Going into the next Matcont release
- Developed new network tomography method
- Designed initial security model for OceanStore
- Served as IEEE 754R secretary
- Responsible for last CLAPACK version
Future Work

- Code development
  - Structural elements and elements for different physics
  - Design and implementation of parallelized version

- Theoretical analysis
  - More damping mechanisms
  - Sensitivity analysis and variational model reduction

- Application collaborations
  - Use of nonlinear effects (quasi-static and dynamic)
  - New designs (e.g. internal dielectric drives)
  - Continued experimental comparisons
Conclusions

- RF MEMS are a great source of problems
  - Interesting applications
  - Interesting physics (and not altogether understood)
  - Interesting numerical mathematics

http://www.cs.berkeley.edu/~dbindel
Sources of Damping

- **Fluid damping**
  - Air is a viscous fluid ($Re \ll 1$)
  - Can operate in a vacuum
  - Shown not to dominate in many RF designs

- **Material losses**
  - Low intrinsic losses in silicon, diamond, germanium
  - Terrible material losses in metals

- **Thermoelastic damping**
  - Volume changes induce temperature change
  - Diffusion of heat leads to mechanical loss

- **Anchor loss**
  - Elastic waves radiate from structure
Sources of Damping

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- Anchor loss
  - Elastic waves radiate from structure
Fabrication Outline

1. Si wafer
2. Deposit 2 microns SiO₂
3. Pattern and etch SiO₂ layer
4. Deposit 2 microns polycrystalline Si
5. Pattern and etch Si layer
6. Release etch remaining SiO₂
Fabrication Outline

1. Si wafer
2. Deposit 2 microns SiO2
   - Pattern and etch SiO2 layer
3. Deposit 2 microns polycrystalline Si
4. Pattern and etch Si layer
5. Release etch remaining SiO2
Fabrication Outline

1. Si wafer
2. Deposit 2 microns SiO2
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Fabrication Result

(C. Nguyen, iMEMS 02)
Role of simulation

HiQLab: Modeling RF MEMS
- Explore fundamental device physics
  - Particularly details of damping
- Detailed finite element modeling
- Reduced models eventually to go into SUGAR

SUGAR: “Be SPICE to the MEMS world”
- Fast enough for early design stages
- Simple enough to attract users
- Support design, analysis, optimization, synthesis
- Verify models by comparison to measurement
Why simulate?

- “Build and break” is too expensive
  - Wafer processing costs months, thousands of dollars
  - Fabrication is imprecise
  - Days or weeks to take good measurements

- Good experiments need good hypotheses
- Even when device behavior is understood, still need to understand system behavior
Computer can assist at many levels:

- Fundamental physics
- Detailed device models
- System-level models and macromodels
- Metrology
- Design optimization
- Design synthesis
Research thrusts

- Model development (e.g. new finite elements)
- Numerical algorithms (e.g. model reduction)
- Numerical software engineering (SUGAR, HiQLab)
- Metrology and comparison to measurement
- Optimization and design synthesis
## Research group

<table>
<thead>
<tr>
<th>Faculty</th>
<th>Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Agogino (ME)</td>
<td>D. Bindel (CS)</td>
</tr>
<tr>
<td>Z. Bai (Math,CS,UCD)</td>
<td>J.V. Clark (AS&amp;T)</td>
</tr>
<tr>
<td>J. Demmel (Math,CS)</td>
<td>C. Cobb (ME)</td>
</tr>
<tr>
<td>S. Govindjee (CEE)</td>
<td>D. Garmire (CS)</td>
</tr>
<tr>
<td>R. Howe (EE,ME)</td>
<td>T. Koyama (CEE)</td>
</tr>
<tr>
<td>K.S.J. Pister (EE)</td>
<td>J. Nie (Math)</td>
</tr>
<tr>
<td>C. Sequin (CS)</td>
<td>H. Wei (CEE)</td>
</tr>
<tr>
<td></td>
<td>Y. Zhang (CEE)</td>
</tr>
</tbody>
</table>
Goal: “Be SPICE to the MEMS world”

- Fast enough for early design stages
- Simple enough to attract users
- Support design, analysis, optimization, synthesis
- Verify models by comparison to measurement

SUGAR

System assembly

- Models
  - Solvers
- Netlist
- Matlab Web Library Interfaces

Results

- Transient analysis
- Steady-state analysis
- Static analysis
- Sensitivity analysis
SUGAR: Analysis of a micromirror

(Mirror design by M. Last)
SUGAR: Design synthesis
SUGAR: Comparison to measurement
Elastic PMLs

\[ \int_\Omega \epsilon(w) : C : \epsilon(u) \, d\Omega - \omega^2 \int_\Omega \rho w \cdot u \, d\Omega = \int_\Gamma w \cdot t_n \, d\Gamma \]

\[ \epsilon(u) = \left( \frac{\partial u}{\partial \tilde{x}} \right)^s \]

- Start from standard weak form
- Introduce transformed \( \tilde{x} \) with \( \frac{\partial \tilde{x}}{\partial x} = \Lambda \)
- Map back to reference system \( (J_\Lambda = \det(\Lambda)) \)
- All terms are symmetric in \( w \) and \( u \)
Elastic PMLs

\[
\int_{\tilde{\Omega}} \tilde{\epsilon}(w) : C : \tilde{\epsilon}(u) \, d\tilde{\Omega} - \omega^2 \int_{\tilde{\Omega}} \rho w \cdot u \, d\tilde{\Omega} = \int_{\Gamma} w \cdot t_n \, d\Gamma
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\[
\tilde{\epsilon}(u) = \left( \frac{\partial u}{\partial \tilde{x}} \right)^s = \left( \frac{\partial u}{\partial x} \Lambda^{-1} \right)^s
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\int_{\Omega} \tilde{\epsilon}(w) : C : \tilde{\epsilon}(u) J_{\Lambda} \, d\Omega - \omega^2 \int_{\Omega} \rho w \cdot u \, J_{\Lambda} \, d\Omega = \int_{\Gamma} w \cdot \hat{t} \, d\Gamma
\]

\[
\tilde{\epsilon}(u) = \left( \frac{\partial u}{\partial \tilde{x}} \right)^s = \left( \frac{\partial u}{\partial x} \Lambda^{-1} \right)^s
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Elastic PMLs

\[ \int_{\Omega} \tilde{\varepsilon}(w) : C : \tilde{\varepsilon}(u) J_{\Lambda} \, d\Omega - \omega^2 \int_{\Omega} \rho w \cdot u J_{\Lambda} \, d\Omega = \int_{\Gamma} \rho w \cdot t_n \, d\Gamma \]

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- Map back to reference system (\( J_{\Lambda} = \text{det}(\Lambda) \))
- All terms are symmetric in \( w \) and \( u \)
Continuum 2D model problem

\[
\lambda(x) = \begin{cases} 
1 - i/\beta \, |x - L|^p, & x > L \\
1, & x \leq L.
\end{cases}
\]

\[
\frac{1}{\lambda} \frac{\partial}{\partial x} \left( \frac{1}{\lambda} \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0
\]
Continuum 2D model problem

\[ \lambda(x) = \begin{cases} 
1 - i \beta |x - L| \rho, & x > L \\
1 & x \leq L
\end{cases} \]

\[
\frac{1}{\lambda} \frac{\partial}{\partial x} \left( \frac{1}{\lambda} \frac{\partial u}{\partial x} \right) - k_y^2 u + k^2 u = 0
\]
Continuum 2D model problem

\[ \lambda(x) = \begin{cases} 
1 - i\beta|\lambda||x - L|^p, & x > L \\
1, & x \leq L.
\end{cases} \]

\[
\frac{1}{\lambda} \frac{\partial}{\partial x} \left( \frac{1}{\lambda} \frac{\partial u}{\partial x} \right) + k_x^2 u = 0
\]

1D problem, reflection of \(O(e^{-k_x \gamma})\)
Discrete 2D model problem

- Discrete Fourier transform in \( y \)
- Solve numerically in \( x \)
- Project solution onto infinite space traveling modes
- Extension of Collino and Monk (1998)
Nondimensionalization

\[ \lambda(x) = \begin{cases} 
1 - i\beta|x - L|^p, & x > L \\
1, & x \leq L.
\end{cases} \]

Rate of stretching: \( \beta h^p \)

Elements per wave: \( (k_x h)^{-1} \) and \( (k_y h)^{-1} \)

Elements through the PML: \( N \)
\[ \lambda(x) = \begin{cases} 
1 - i\beta|x - L|^p, & x > L \\
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\end{cases} \]

Rate of stretching: \( \beta h^p \)

Elements per wave: \( (k_x h)^{-1} \) and \( (k_y h)^{-1} \)

Elements through the PML: \( N \)
Discrete reflection behavior

\[ - \log_{10}(r) \text{ at } (k h)^{-1} = 10 \]

Number of PML elements

\[ \log_{10}(\beta h) \]

\[ -\log_{10}(r) \text{ at } (kh)^{-1} = 10 \]

Quadratic elements, \( p = 1, (k_x h)^{-1} = 10 \)
Model discrete reflection as two parts:

- **Far-end reflection (clamping reflection)**
  - Approximated well by continuum calculation
  - Grows as $(k_x h)^{-1}$ grows

- **Interface reflection**
  - Discrete effect: mesh does not resolve decay
  - Does not depend on $N$
  - Grows as $(k_x h)^{-1}$ shrinks
Discrete reflection behavior

- $-\log_{10}(r) \text{ at } (kh)^{-1} = 10$
- $-\log_{10}(r_{\text{interface}} + r_{\text{nominal}}) \text{ at } (kh)^{-1} = 10$

Number of PML elements

Quadratic elements, $p = 1$, $(k_x h)^{-1} = 10$

- Model does well at predicting actual reflection
- Similar picture for other wavelengths, element types, stretch functions
Choosing PML parameters

- Discrete reflection dominated by
  - Interface reflection when \( k_x \) large
  - Far-end reflection when \( k_x \) small

- Heuristic for PML parameter choice
  - Choose an acceptable reflection level
  - Choose \( \beta \) based on interface reflection at \( k_x^{\text{max}} \)
  - Choose length based on far-end reflection at \( k_x^{\text{min}} \)
Thermoelastic damping (TED)

\[ u \text{ is displacement and } T = T_0 + \theta \text{ is temperature} \]

\[ \sigma = C \varepsilon - \beta \theta \mathbf{1} \]

\[ \rho u_{tt} = \nabla \cdot \sigma \]

\[ \rho c_v \theta_t = \nabla \cdot (\kappa \nabla \theta) - \beta T_0 \text{tr}(\varepsilon_t) \]

- Volumetric strain rate drives energy transfer from mechanical to thermal domain
  - Irreversible diffusion \( \Longrightarrow \) mechanical damping
  - Not often an important factor at the macro scale
  - Recognized source of damping in microresonators
- Zener: semi-analytical approximation for TED in beams
- We consider the fully coupled system
Nondimensionalization

\[
\begin{align*}
\sigma &= \hat{C}\epsilon - \xi \theta 1 \\
u_{tt} &= \nabla \cdot \sigma \\
\theta_t &= \eta \nabla^2 \theta - \text{tr}(\epsilon_t)
\end{align*}
\]

\[
\xi := \left(\frac{\beta}{\rho c}\right)^2 \frac{T_0}{c_v} \text{ and } \eta := \frac{\kappa}{\rho c_v c L}
\]

Length \sim L \\
Time \sim L/c, \text{ where } c = \sqrt{E/\rho} \\
Temperature \sim T_0 \frac{\beta}{\rho c_v}
Scaling analysis

\[ \sigma = \hat{C}\epsilon - \xi \theta 1 \]

\[ u_{tt} = \nabla \cdot \sigma \]

\[ \theta_t = \eta \nabla^2 \theta - \text{tr}(\epsilon_t) \]

\[ \xi := \left( \frac{\beta}{\rho c} \right)^2 \frac{T_0}{c_v} \text{ and } \eta := \frac{\kappa}{\rho c_v c_L} \]

- Micron-scale poly-Si devices: \( \xi \) and \( \eta \) are \( \sim 10^{-4} \).
- Small \( \eta \) leads to thermal boundary layers
- Linearize about \( \xi = 0 \)
Discrete mode equations

\[
\sigma = \hat{C}\epsilon - \xi \theta 1
\]

\[
u_{tt} = \nabla \cdot \sigma
\]

\[
\theta_t = \eta \nabla^2 \theta - \text{tr}(\epsilon_t)
\]

\[
\sigma = \hat{C}\epsilon - \xi \theta 1
\]

\[
-\omega^2 u = \nabla \cdot \sigma
\]

\[
i\omega \theta = \eta \nabla^2 \theta - i\omega \text{tr}(\epsilon)
\]

\[
-\omega^2 M_{uu} u + K_{uu} u + K_{ut} \theta = 0
\]

\[
i\omega D_{tt} \theta + K_{tt} \theta + i\omega D_{tu} u = 0
\]
Perturbation computation

\[-\omega^2 M_{uu} u + K_{uu} u + K_{ut} \theta = 0\]
\[i \omega D_{tt} \theta + K_{tt} \theta + i \omega D_{tu} u = 0\]

Approximate \( \omega \) by perturbation about \( K_{ut} = 0 \):

\[-\omega_0^2 M_{uu} u_0 + K_{uu} u_0 = 0\]
\[i \omega_0 D_{tt} \theta_0 + K_{tt} \theta_0 + i \omega_0 D_{tu} u_0 = 0\]

Choose \( v : v^T u_0 \neq 0 \) and compute

\[
\begin{bmatrix}
-\omega_0^2 M_{uu} + K_{uu} & -2\omega_0 M_{uu} u_0 \\

v^T & 0
\end{bmatrix}
\begin{bmatrix}
\delta u \\
\delta \omega
\end{bmatrix}
= 
\begin{bmatrix}
-K_{ut} \theta_0 \\
0
\end{bmatrix}
\]
Comparison to Zener’s model

- Comparison of fully coupled simulation to Zener approximation over a range of frequencies
- Real and imaginary parts after first-order correction agree to about three digits with Arnoldi
Thermoelastic boundary layer

- One-dimensional test problem (longitudinal mode in a bar)
- Fixed temperature and displacement at left
- Free at right
Shear ring resonator

- Ring is driven in a shearing motion
- Can couple ring to other resonators
- How do we track the desired mode?
Mode tracking

Find a continuous solution to

\[
\left( K(s) - \omega(s)^2 M(s) \right) u(s) = 0.
\]

- \( K \) and \( M \) are symmetric and \( M > 0 \)
- Eigenvectors are \( M \)-orthogonal
- Perturbation theory gives good shifts
- Look if \( u(s + h) \) and \( u(s) \) are on the same path by looking at \( u(s + h)^T M(s + h) u(s) \)
- Many more subtleties in the nonsymmetric case
  - Focus of the \textit{CIS algorithm}
Mode tracking in a shear resonator