Continuation of Invariant Subspaces of Sparse Parameter-Dependent Matrices

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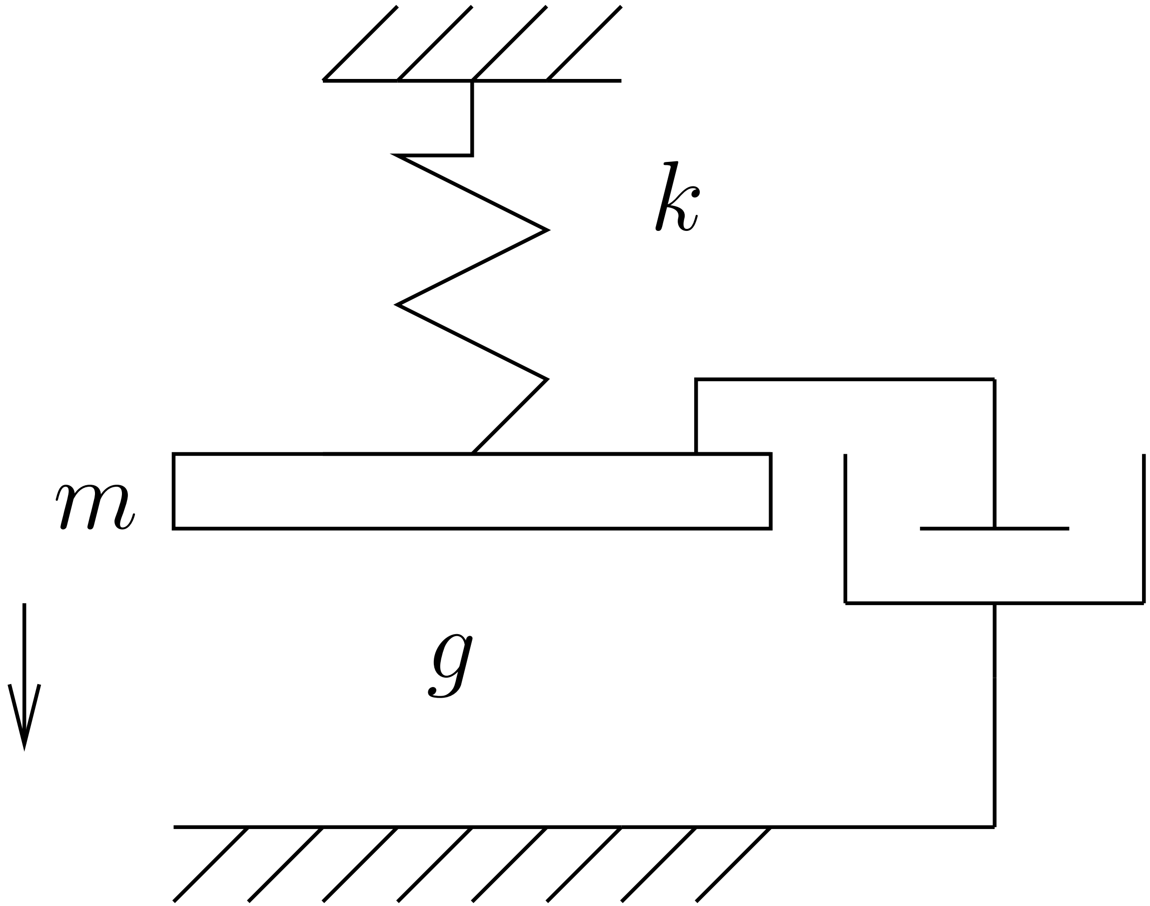
Outline

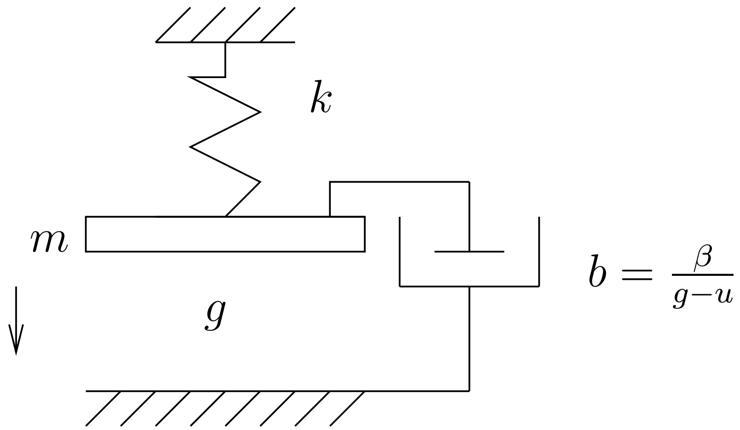
- Why parameter-dependent eigenproblems?
- The sparse CIS procedure
- Conclusions and future work

Example: Gap pull-in



$$b = \frac{\beta}{g-u}$$

Example: Gap pull-in



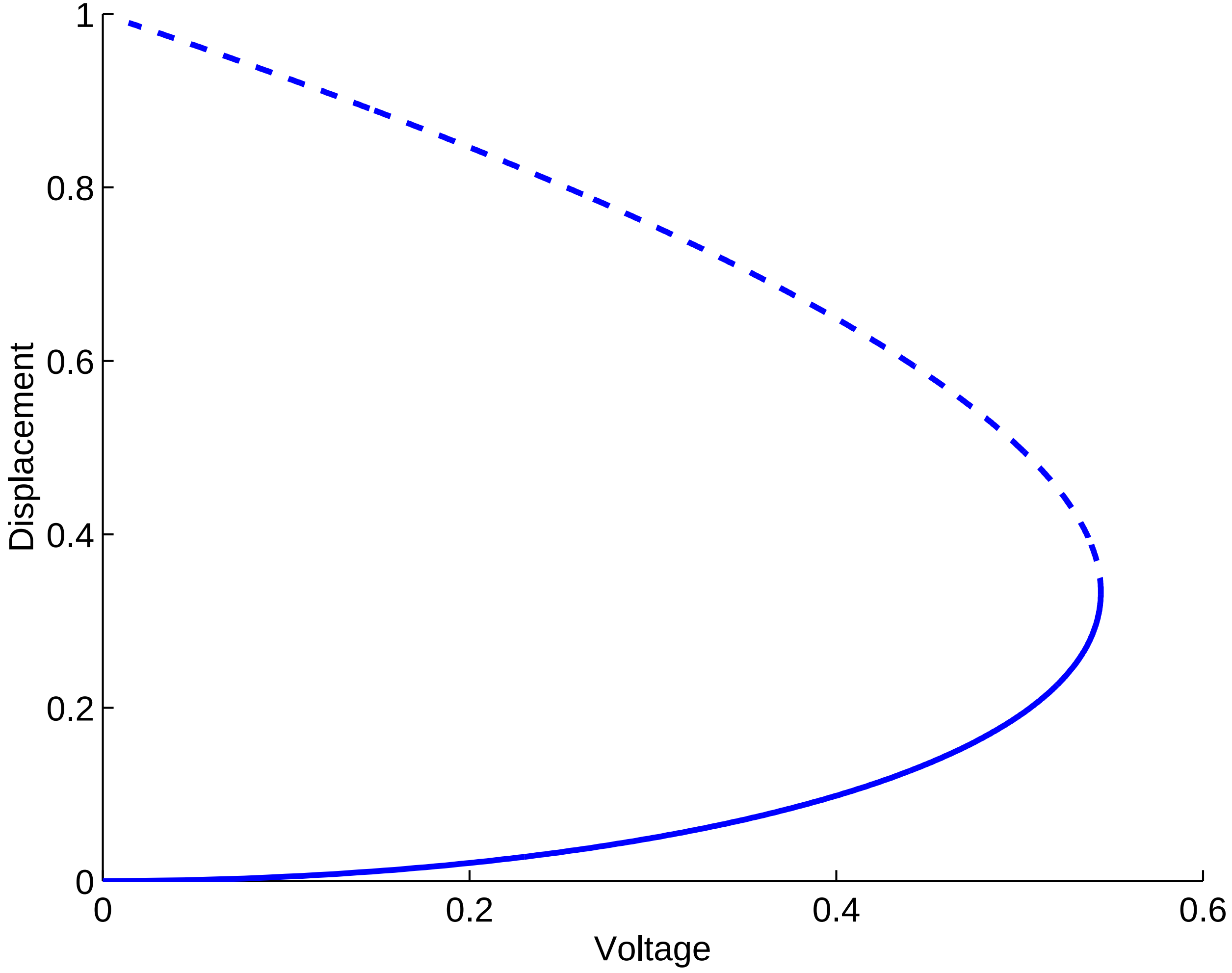
Nonlinear governing equation:

$$mu'' + \frac{\beta}{g-u}u' + ku - \frac{\alpha V^2}{2(g-u)^2} = 0$$

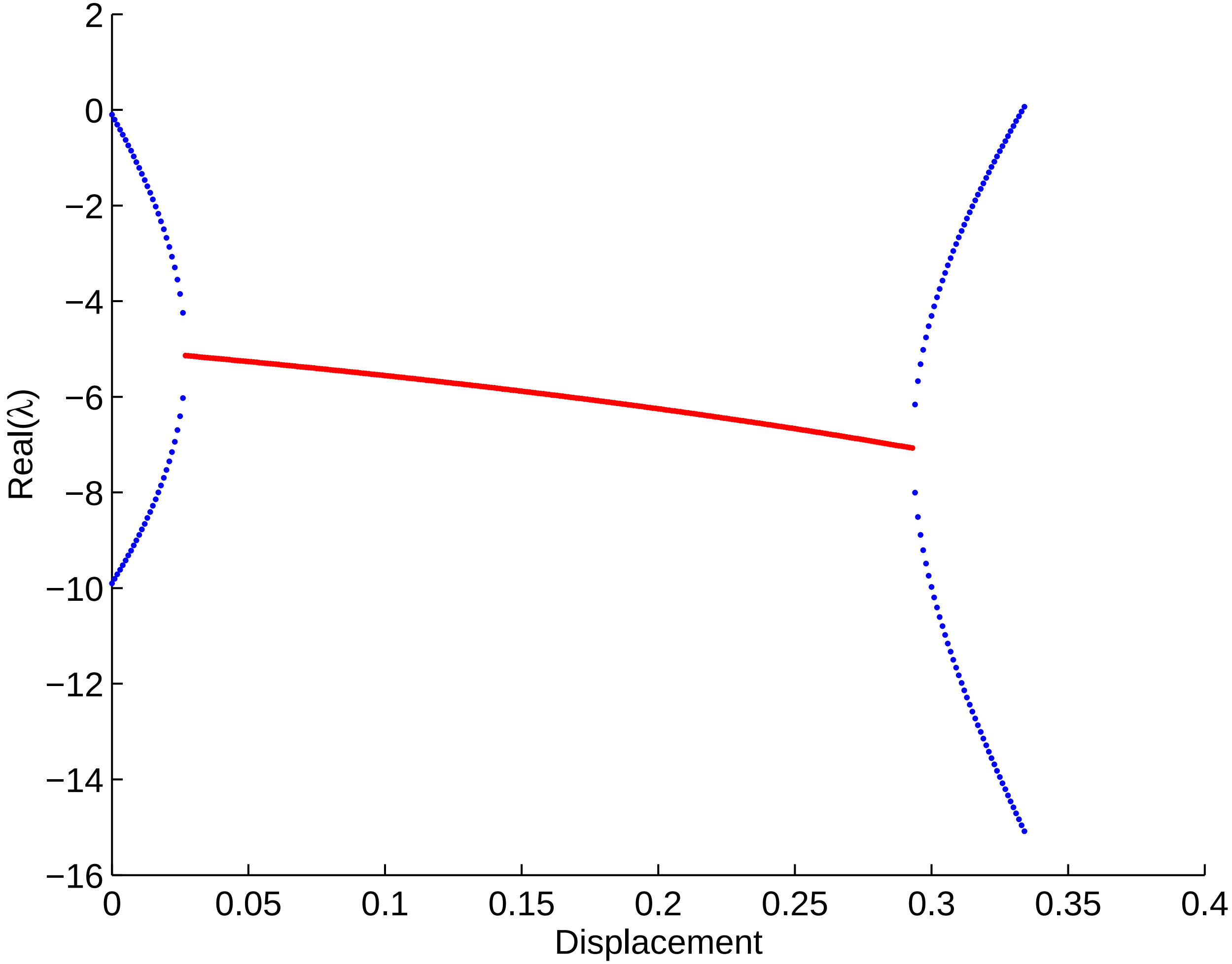
Linearized at equilibrium $ku - \frac{\alpha V^2}{2(g-u)^2} = 0$:

$$m(\delta u)'' + \frac{\beta}{g-u}(\delta u)' + k\left(1 - \frac{2u}{g-u}\right)(\delta x) = 0$$

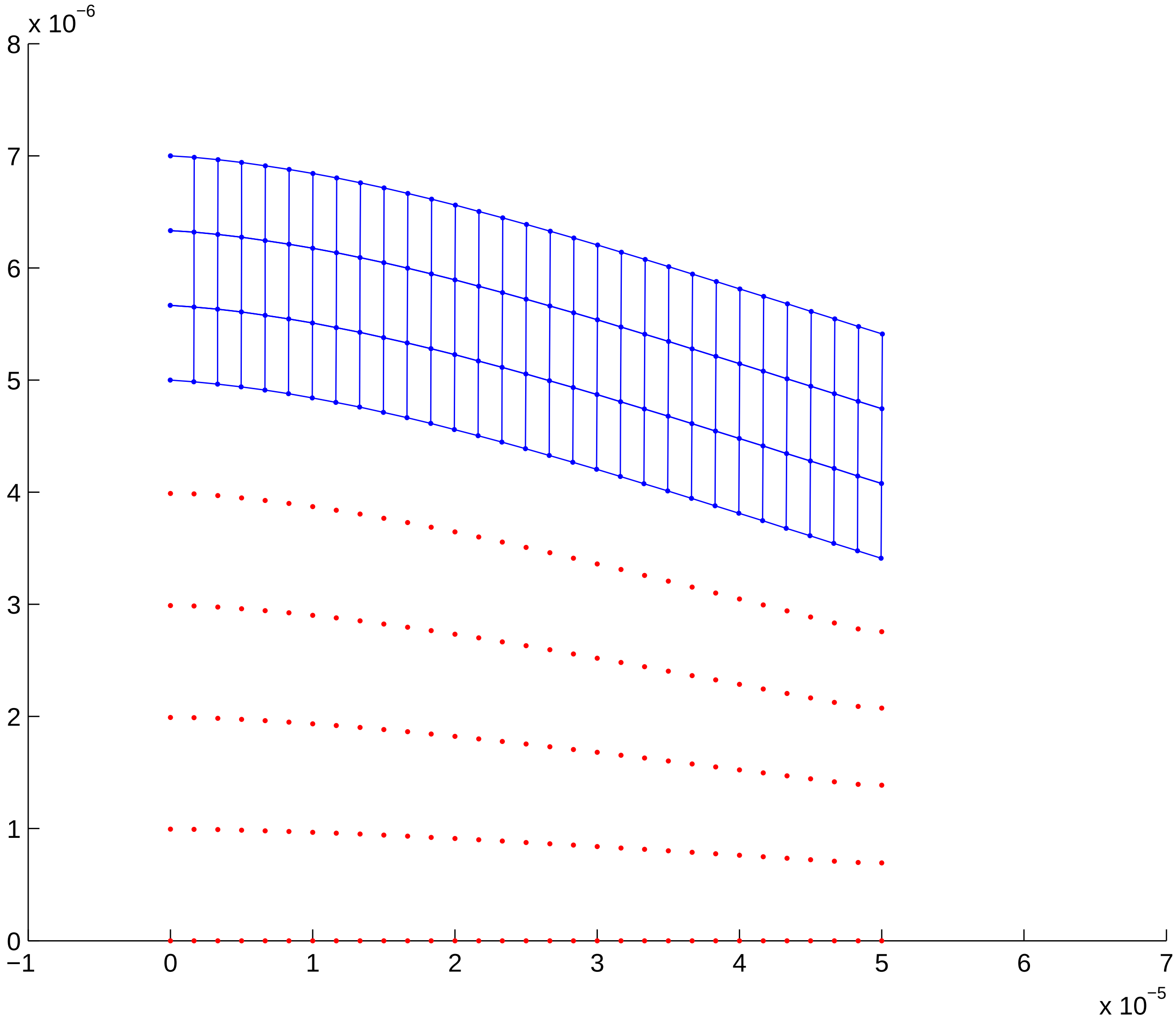
Example: Gap pull-in



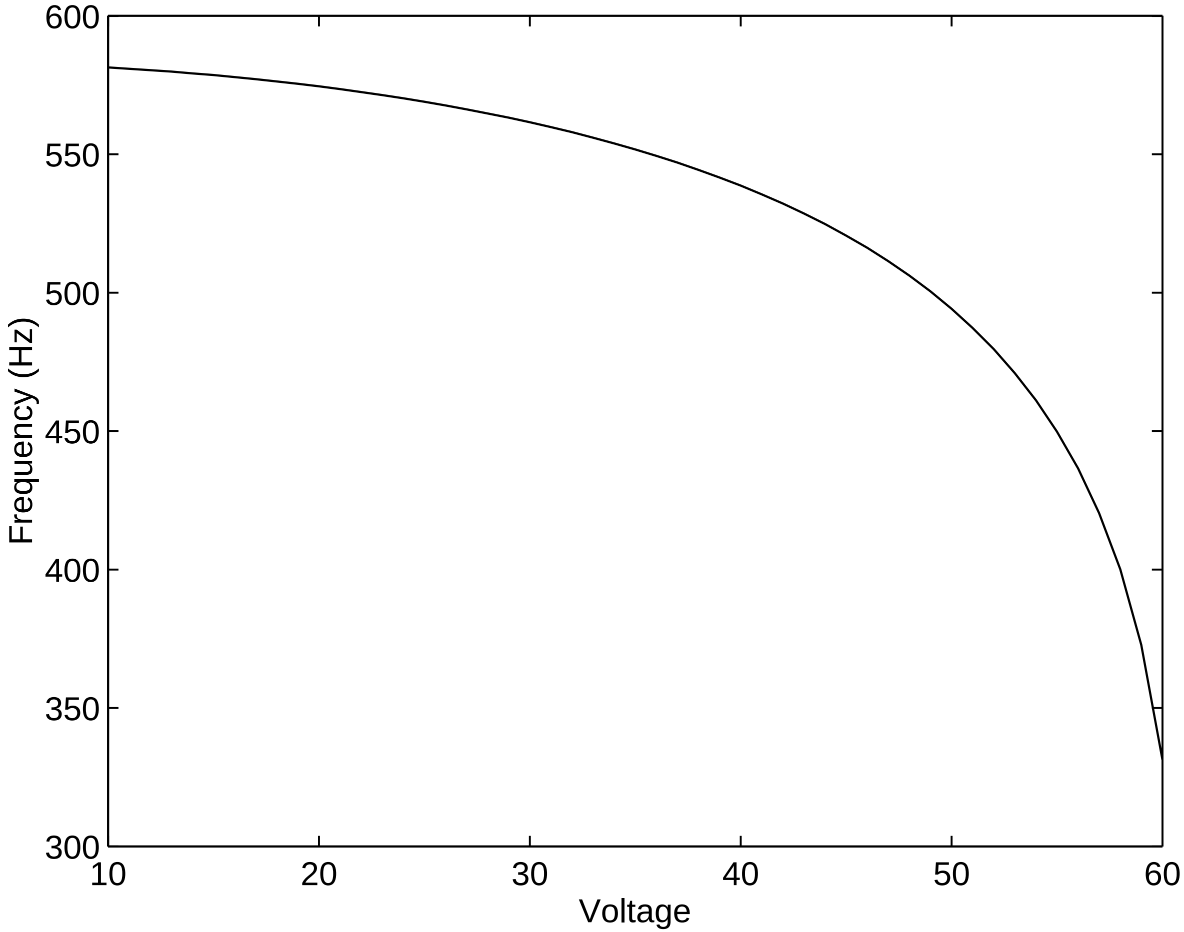
Example: Gap pull-in



Example: Cantilever tuning



Example: Cantilever tuning



General picture

Given the ODE

$$\frac{du}{dt} = f(u, \alpha)$$

write a branch of equilibria $(u(s), \alpha(s))$ so that $f(u(s), \alpha(s)) = 0$. Analyze stability from eigenvalues of

$$A(s) = D_u f(u(s), \alpha(s)).$$

For differentiable $A : [0, 1] \to \mathbb{R}^{n \times n}$:

- Eigenvalues are continuous $\lambda_i(s)$
- $\lambda_i(s)$ differentiable when distinct
- For a distinct set, there is a differentiable subspace

Related work

- Moving frames on solution manifolds
  (Rheinboldt)
- Real-analytic SVD computation
  (Bunse-Gerstner, Byers, Mehrmann, Nichols)
- Real-analytic null space computations and DAEs
  (Kunkel, Mehrmann)
- Smooth matrix decompositions
  (Dieci, Eirola)
- Bifurcation analysis
  (Thummler, Beyn, Kless; Friedman, Dieci, Demmel)
- Perturbation theory
  (Kato; Stewart; Demmel)

CIS algorithm

Compute a continuous block Schur form

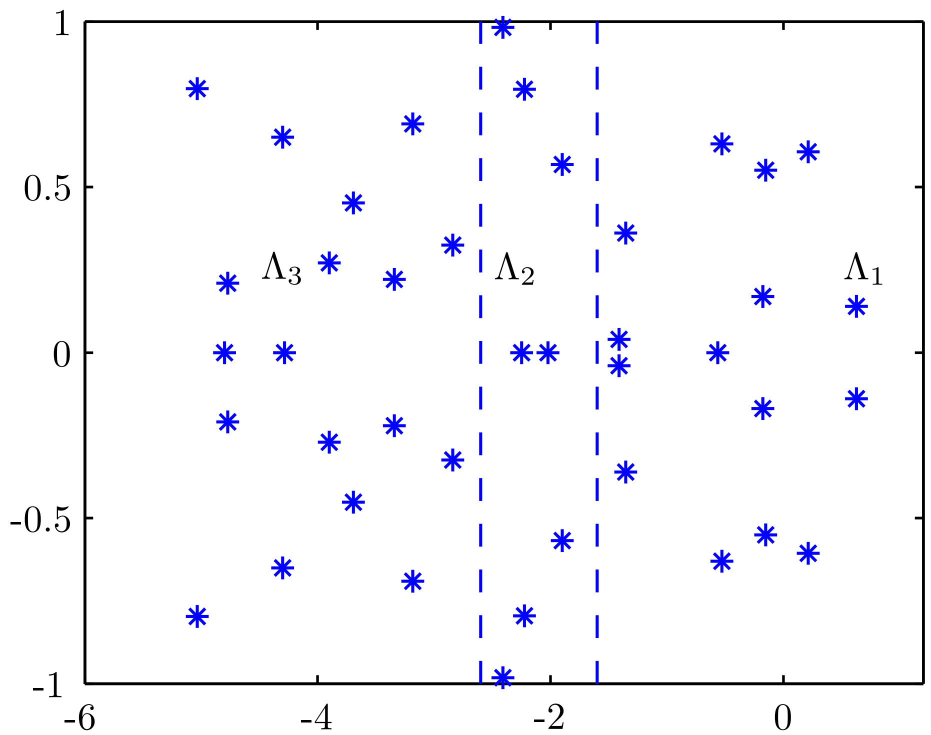
$$A(s) = \begin{bmatrix} Q_1(s) & Q_2(s) \end{bmatrix} \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix} \begin{bmatrix} Q_1(s) & Q_2(s) \end{bmatrix}^T$$

- Components:
  - Choose initial invariant subspace
  - Continue one step (predictor/corrector)
  - Normalize the solution
  - Adapt space and step size to improve convergence, resolve features of interest

Sparse case

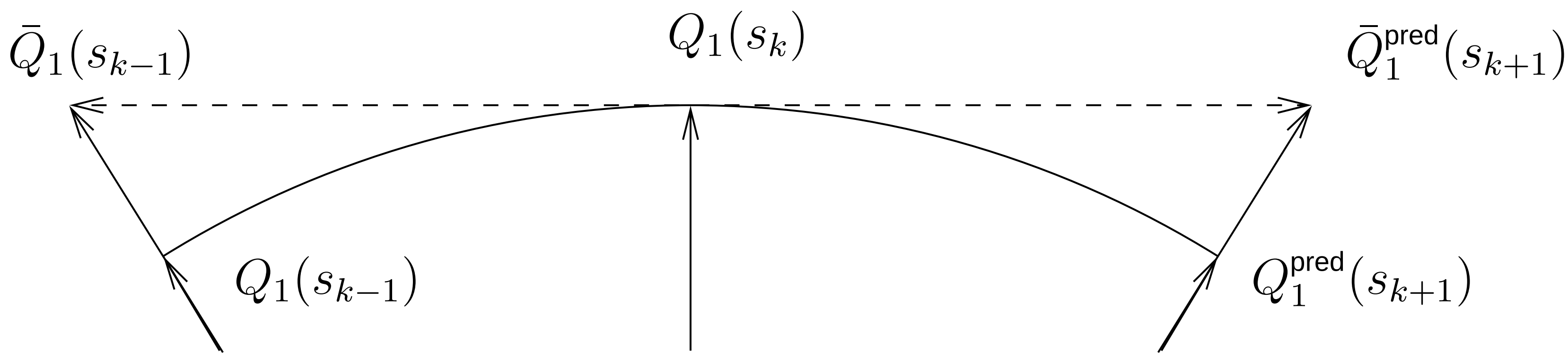
- Compute only $Q_1(s)$
- Choose from a projection space $\mathcal{V}$
  - Changes from step to step
  - Should contain desired space + a little bit
  - Often Krylov or block Krylov
  - May use information from an outer iteration
  - May patch together bases from several steps

Initialization



- Compute rightmost part of the spectrum
- Include all unstable eigenvalues + a few stable ones
- Keep eigenvalue clusters together
  (prevent artificially short steps)

Prediction



- Normalize to tangent plane:

$$\bar{Q}_1(s) = Q_1(s)\left(Q_1(s_k)^T Q_1(s)\right)^{-1}$$

- Predict $\bar{Q}_1(s_{k+1})$ by polynomial fitting through $\bar{Q}_1(s_k), \bar{Q}_1(s_{k-1}), \ldots$.
- Suggests projection space should include computed spaces from previous few steps.

Correction

- Two variants of Newton:
  - Iterate on full system (Beyn, Thummler, Kless); linearization is a bordered Sylvester equation
  - Eliminate T_{11} to get an algebraic Riccatti eq (Demmel); linearization is an ordinary Sylvester equation
- Sparse case: restrict $\operatorname{span}(\hat{Q}) \subset \mathcal{V}$ and apply Galerkin
  - Can eliminate to get a Riccati equation again
  - Some changes if $\operatorname{span}(Q(s)) \not\subset \mathcal{V}$

Correction

Seek $R\left(\bar{Q}_1(s_{k+1}), \bar{T}_{11}(s_{k+1})\right) = 0$, where

$$R = \begin{bmatrix} A(s_{k+1})\bar{Q}(s_{k+1}) - \bar{Q}_1(s_k + 1)\bar{T}_{11}(s_{k+1}) \\ Q_1(s_k)^T\bar{Q}_1(s_{k+1}) - I \end{bmatrix}$$

Or write $\bar{A}(s) = Q(s_k)^T A(s) Q(s_k)$ so that

$$\begin{bmatrix} \bar{A}_{11}(s) & \bar{A}_{12}(s) \\ \bar{A}_{21}(s) & \bar{A}_{22}(s) \end{bmatrix} \begin{bmatrix} I \\ Y(s) \end{bmatrix} = \begin{bmatrix} I \\ Y(s) \end{bmatrix} \bar{T}_{11}(s)$$

and eliminate $\bar{T}_{11}(s)$ and solve $F(Y) = 0$ at s_{k+1}, where

$$F(Y) = \bar{A}_{22}Y - Y\bar{A}_{11} + \bar{A}_{21} - Y\bar{A}_{12}Y$$

Acceptance criteria

- Diagnostics quantities: canonical angles, number of Newton iterations
- Start from standard perturbation result:

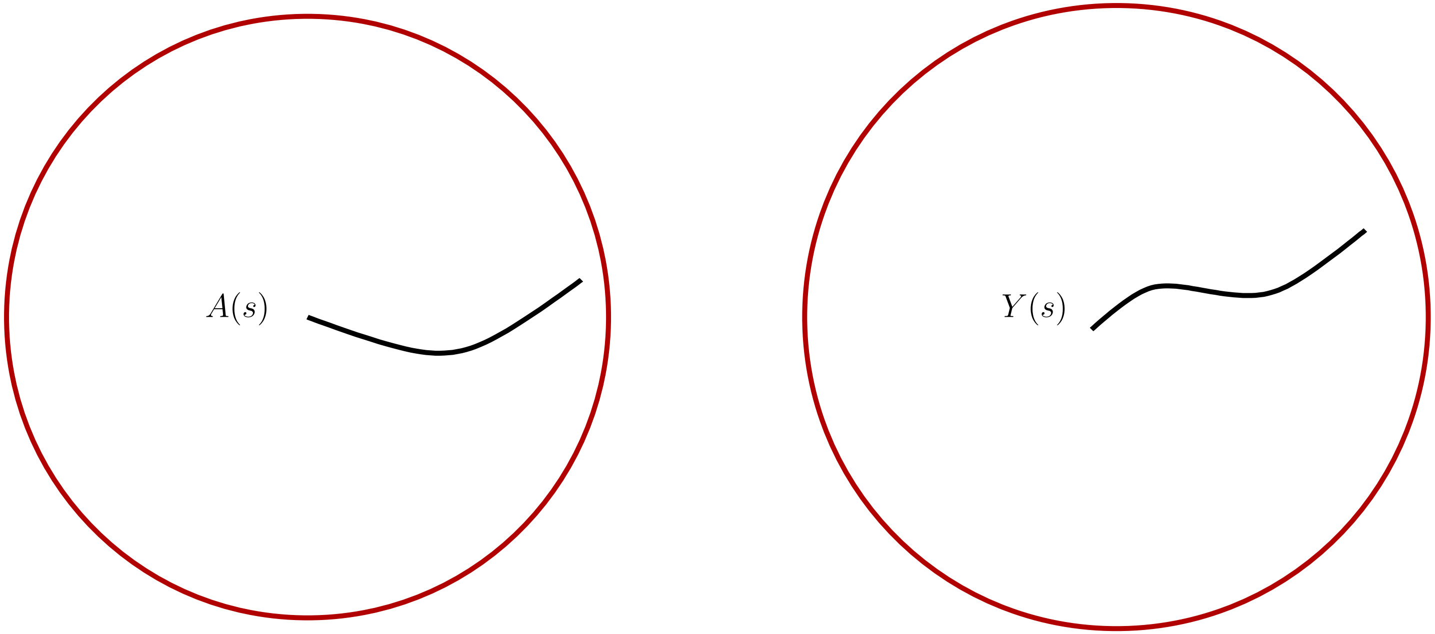
$$\kappa := \frac{\|\bar{A}_{12}\|_2 \|F(Y_0)\|_F}{\text{sep}(\bar{A}_{11} + \bar{A}_{12}Y_0, \bar{A}_{22} - Y_0\bar{A}_{12})^2}$$

  - Unique smallest-norm solution Y when $\kappa < 1/4$
  - Newton converges from Y_0 when $\kappa < 1/12$
  - Forcing $\kappa < 1/12$ at s_{k+1} is too severe
  - Can we do better?

Perturbation approaches

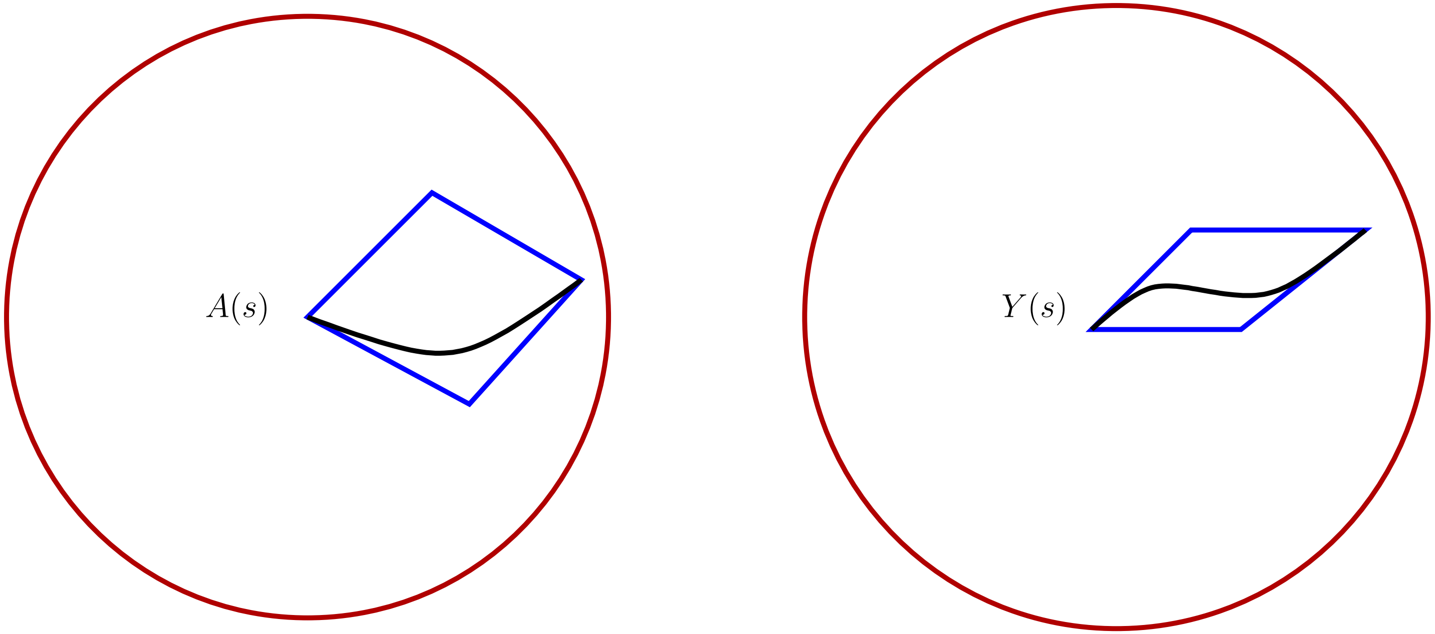
$A(s)$ and $Y(s)$ trace some paths in matrix spaces.

Perturbation approaches



Can determine that if $A(s)$ stays in some region, $Y(s)$ is well defined and stays in some other region.

Perturbation approaches



Can we test with smaller regions?

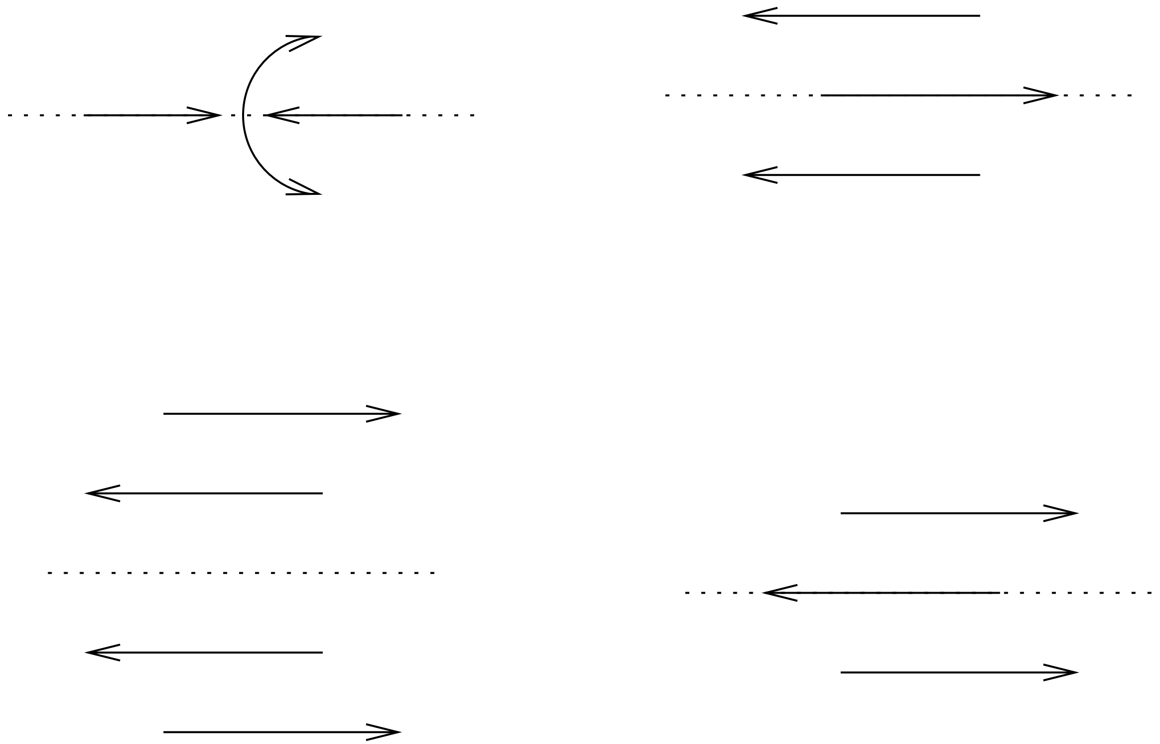
Test for a connecting space

Given Y^{end}, is there a differentiable solution $Y(s)$ between such that $Y(s_k) = 0$ and $Y(s_{k+1}) = Y^{\mathrm{end}}$?

Test by interpolation:

- Stewart's perturbation result for fixed matrices extends to parameter-dependent matrices
- Linearly interpolate $\bar{A}(s)$ between s_k and s_{k+1}
- Linearly interpolate starting guess $Y_0(s)$ between $Y(s_k) = 0$ and $Y(s_{k+1})$
- Test if $\kappa(s) < 1/4$ uniformly

Features



May need to adjust space if

- Real parts of continued eigenvalues overlap the rest of the spectrum (generic possibilities shown)
- Eigenvalues cross imaginary axis (bifurcation)

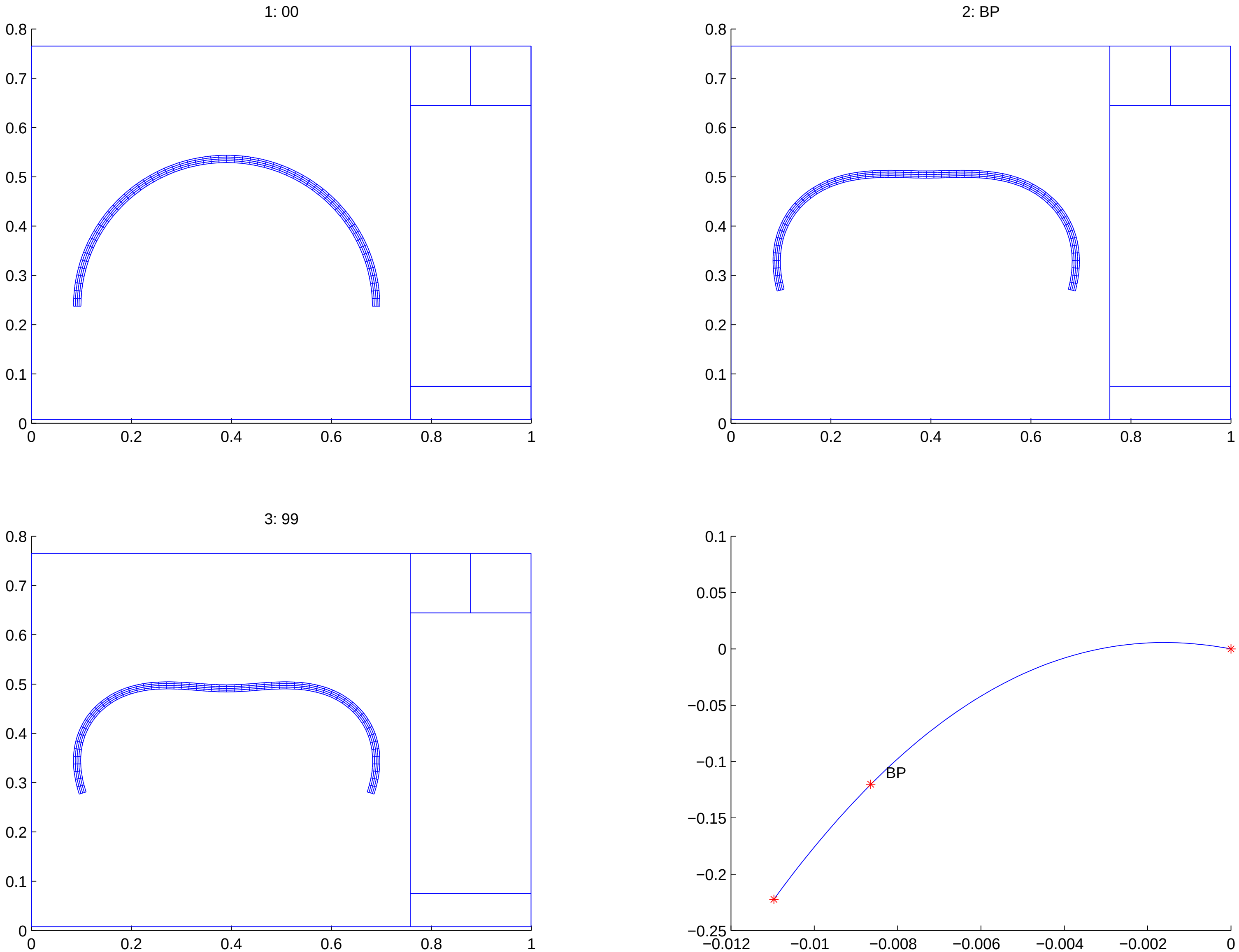
Global control

- Attempt a step from s_k to s_{k+1}
- If convergence failure,
  - Change the step size and retry, or
  - Choose a different space at s_k and retry
- Check for interesting features (bifurcation or overlap)
  - If several occur, cut step to resolve them
  - If one occurs, may reinitialize at s_{k+1}

Use in MATCONT-L

- Integrated sparse CIS into the MATCONT bifurcation analysis tool (Dhooge, Govaerts, Kuznetsov, Mestrom, Riet)
- Replaced test functions on $A(s)$ with functions on $T_{11}(s)$
- Changes to MATCONT routines for step-size control, test function evaluation, and location of bifurcations

Example: Arch snap-through



Conclusions and future work

- Discussed invariant subspace continuation and some applications for bifurcation analysis
- Future work:
  - Smarter projection space formation
  - Structure preservation for second order systems
  - Eigencontinuation for PDAE discretizations