Some Eigenvalue Computations: A Whirlwind Tour of Numerics

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How would you solve it?

You need the eigenvalues of a 20-by-20 matrix A.
The Mathematica license server is down.

What do you do?

First attempt

The characteristic polynomial of A is

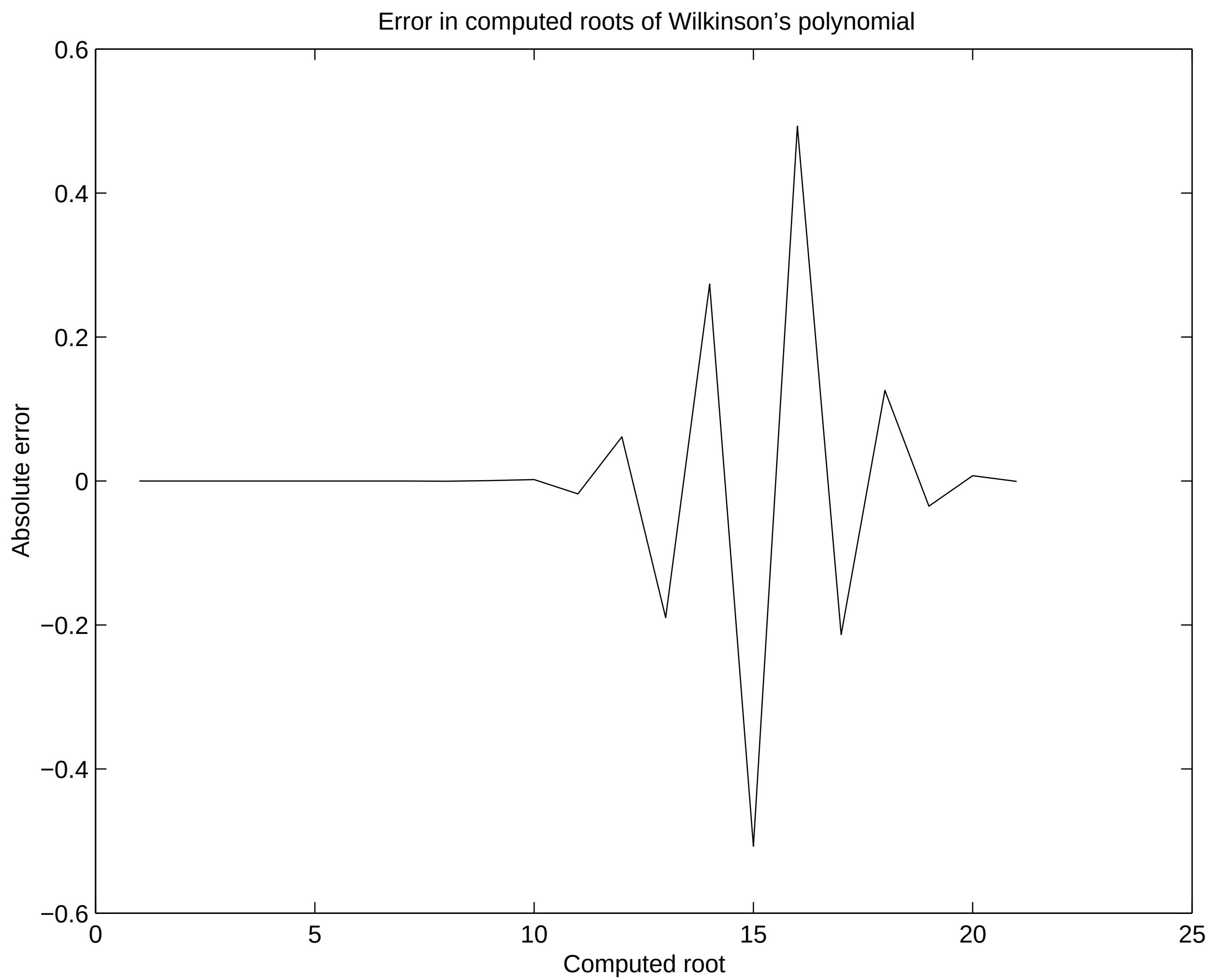
$$p(\lambda) = \det(A - \lambda I).$$

Could we solve the equation $p(\lambda) = 0$?

Wilkinson's example

Suppose $p(\lambda) = \prod_{i=1}^{21}(x - i)$.

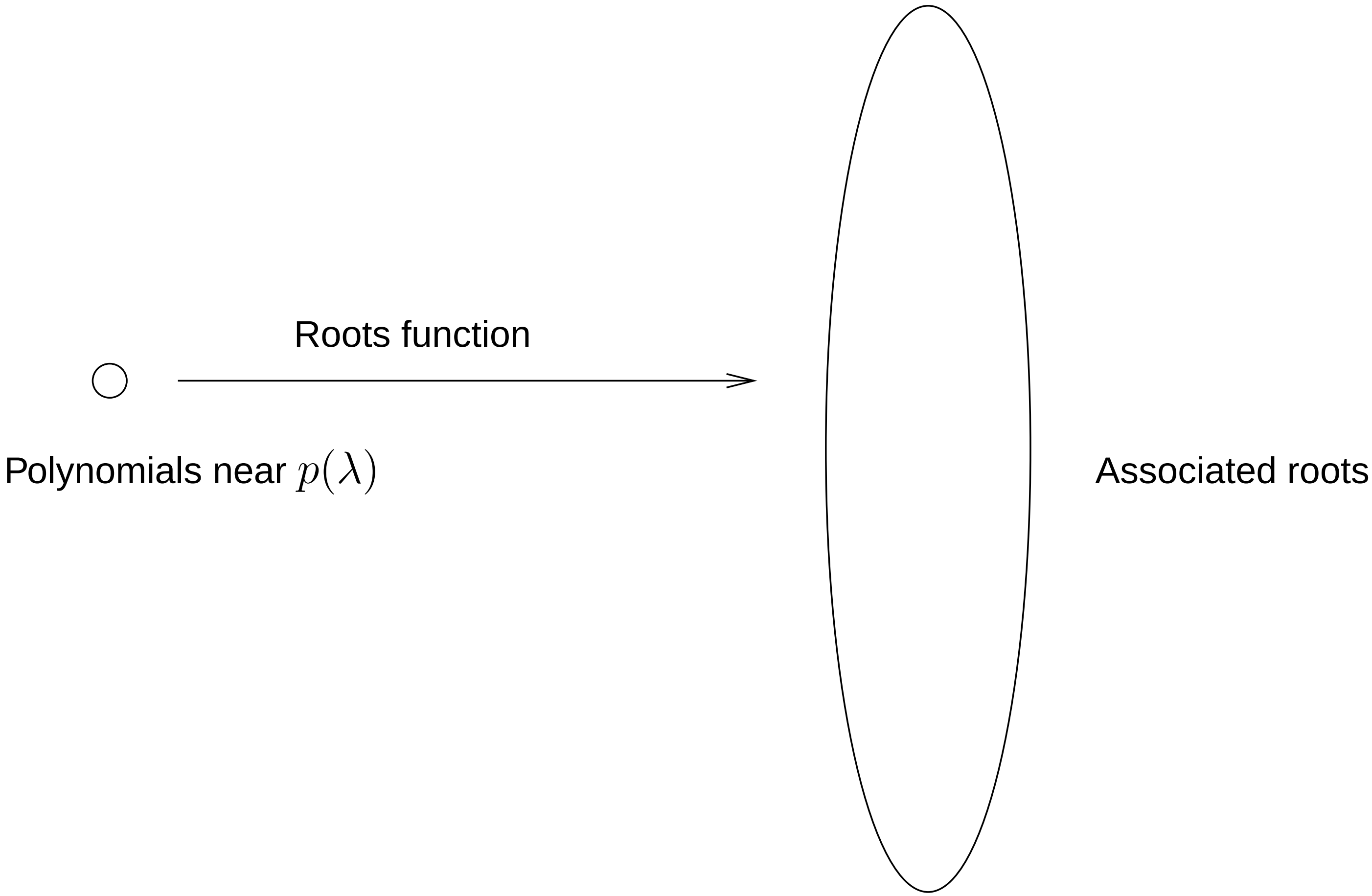
Compute the coefficients and try to recover the roots.



What went wrong?

- I rounded the coefficients to double precision (16 decimal digits)
- My error would be little better even if I committed no additional rounding errors.
- The roots are very sensitive functions of the coefficients.
- But the eigenvalues of a symmetric matrix are not so sensitive to the *matrix* coefficients!

Problem sensitivity



How can I quantify the sensitivity of the problem?

Problem sensitivity

The *condition number* κ measures sensitivity of the roots (λ) to changes in polynomial coefficients (p):

$$\frac{||\delta\lambda||}{||\lambda||} \leq \kappa \frac{||\delta p||}{||p||} + \text{higher order terms}$$

$$\kappa = \left\| \frac{\partial \lambda}{\partial p} \right\| \frac{||p||}{||\lambda||}$$

If κ is very large, we call the problem *ill-conditioned*.

The same idea works for other problems.

Forward and backward error

Forward error: How wrong did I get the answer?

$$\lambda_{\text{computed}} = \lambda_{\text{exact}}(A) + \delta\lambda$$

Backward error: How wrong did I get the problem?

$$\lambda_{\text{computed}} = \lambda_{\text{exact}}(A + \delta A)$$

We usually seek algorithms with small backward error.

Forward = sensitivity × backward

We try to design algorithms so that

$$\frac{\text{backward error}}{\text{input value}} \leq C\epsilon$$

where C is some constant depending on the dimension and ϵ is the rounding threshold (around 10^{-16} for double precision). Then to first order

$$\frac{\text{forward error}}{\text{output value}} \leq \kappa \times \frac{\text{backward error}}{\text{input value}} \leq C\kappa\epsilon$$

Sensitivity of eigenvalues