Some Eigenvalue Computations: A Whirlwind Tour of Numerics

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How would you solve it?

You need the eigenvalues of a 20-by-20 matrix $A$. The Mathematica license server is down. What do you do?
The characteristic polynomial of $A$ is

$$p(\lambda) = \det(A - \lambda I).$$

Could we solve the equation $p(\lambda) = 0$?
Wilkinson’s example

Suppose \( p(\lambda) = \prod_{i=1}^{21}(x - i) \).

Compute the coefficients and try to recover the roots.
What went wrong?

- I rounded the coefficients to double precision (16 decimal digits)
- My error would be little better even if I committed no additional rounding errors.
- The roots are very sensitive functions of the coefficients.
- But the eigenvalues of a symmetric matrix are not so sensitive to the matrix coefficients!
Problem sensitivity

How can I quantify the sensitivity of the problem?
The *condition number* $\kappa$ measures sensitivity of the roots ($\lambda$) to changes in polynomial coefficients ($p$):

$$\frac{\| \delta \lambda \|}{\| \lambda \|} \leq \kappa \frac{\| \delta p \|}{\| p \|} + \text{higher order terms}$$

$$\kappa = \left\| \frac{\partial \lambda}{\partial p} \right\| \frac{\| p \|}{\| \lambda \|}$$

If $\kappa$ is very large, we call the problem *ill-conditioned*. The same idea works for other problems.
Forward and backward error

**Forward error:** How wrong did I get the answer?

\[ \lambda_{\text{computed}} = \lambda_{\text{exact}}(A) + \delta \lambda \]

**Backward error:** How wrong did I get the problem?

\[ \lambda_{\text{computed}} = \lambda_{\text{exact}}(A + \delta A) \]

We usually seek algorithms with small backward error.
We try to design algorithms so that

\[
\frac{\text{backward error}}{\text{input value}} \leq C\epsilon
\]

where \(C\) is some constant depending on the dimension and \(\epsilon\) is the rounding threshold (around \(10^{-16}\) for double precision). Then to first order

\[
\frac{\text{forward error}}{\text{output value}} \leq \kappa \times \frac{\text{backward error}}{\text{input value}} \leq C\kappa\epsilon
\]
Sensitivity of eigenvalues