HW 4

You may (and should) talk about problems with each other and with me, providing attribution for any good ideas you might get. Your final write-up should be your own.

1: Contrary conditioning  Given a scalar $C > 1$, find $A \in \mathbb{R}^{2 \times 2}$ for which all eigenvalues are one but $\kappa_2(A) \geq C$.

2: Interesting identity  Suppose $X, Y \in \mathbb{R}^{n \times k}$. Show that if $\lambda \neq 0$ is an eigenvalue of $XY^T$, then
\[
\begin{bmatrix}
-\lambda I & X \\
Y^T & -I
\end{bmatrix}
\]
is singular. Via this formulation, show $\lambda$ is also an eigenvalue of $Y^T X$.

3: Vector recovery  Suppose $T \in \mathbb{R}^{n \times n}$ is upper triangular and $\lambda = t_{ii}$ is a simple eigenvalue of $T$. Give a code to compute a column eigenvector $v$ in $O(n^2)$ time.

1 function [v] = uptri_eigenvec(T, i)

4: Somewhat symmetric  Suppose $A = H + E$ where $H = H^*$. Argue that if $\lambda = \alpha + \beta i$ is an eigenvalue of $A$, then $|\beta| \leq n\|E\|_2$.

5: Real rotations  Suppose $A \in \mathbb{R}^{n \times n}$ has a complex conjugate pair of eigenvalues $\mu \exp(\pm i \theta) = \alpha \pm \beta i$ and corresponding eigenvectures $u \pm vi$, with $\mu$ larger than the magnitude of any other eigenvalue. Show that power iteration from a random starting vector gives the sequence
\[
v_k \approx u \cos(k \theta + \gamma) - v \sin(k \theta + \gamma)
\]
for large $k$.

6: Double-shift iteration  Suppose $A \in \mathbb{R}^{n \times n}$ has a complex conjugate pair of eigenvalues near $\alpha + \beta i$. Without resorting to complex arithmetic, give a two-dimensional variant of Rayleigh quotient iteration that gives a rapidly-convergent estimate for the invariant subspace associated with the pair of eigenvalues. You may wish to build from the codes in the notes first.

1 function [V, L] = rqi2d(A, a, b, rtol)