HW 3
Due Oct 10.

1: Tree climbing  Let $p$ be a parent vector for a tree with nodes $1,\ldots,n$, i.e. $p_i$ is the index of the parent of node $i$ (or zero if $i$ is the root). Assume that $p_i < i$, so that children appear after parents in the ordering. Now, suppose that $A$ is an $n \times n$ matrix such that

$$A_{ij} = \begin{cases} 
\alpha_i, & i = j, \\
\beta_j, & i = p_j, \\
\beta_i, & j = p_i, \\
0, & \text{otherwise},
\end{cases}$$

and that $A$ is positive definite. Write a function (not using the sparse intrinsics in MATLAB) that solves linear systems of the form $Ax = b$ in $O(n)$ time.

Your function should look like

function [x] = p1tree(p, alpha, beta, b);
% Equivalent to
% A = diag(alpha);
% for k = 2:n
% A(k,p(k)) = beta(k);
% A(p(k),k) = beta(k);
% end
% x = A\b;

Alternately, you may implement an equivalent Python routine with an equivalent input/output signature.

2: Modified metrics  Let $M \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and let $\langle x, y \rangle_M \equiv y^T M x$ and $\|x\|_M = \sqrt{\langle x, x \rangle_M}$ be the corresponding induced norm and inner product. $W \in \mathbb{R}^{n \times n}$ is $M$-orthogonal if $W^T M W = M$.

1. Write the normal equations for minimizing $\|Ax - b\|_M^2$.

2. Given $A \in \mathbb{R}^{n \times n}$, write a function to compute $A = WR$, where $W$ is $M$-orthogonal and $R$ is upper triangular. Your function should take the form

function [W,R] = p2wr(A,M)
3: An extended system  Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, be full rank.

1. Show that if
   \[
   \begin{bmatrix}
   I & A \\
   A^T & 0
   \end{bmatrix}
   \begin{bmatrix}
   r \\
   x
   \end{bmatrix} = \begin{bmatrix}
   b \\
   0
   \end{bmatrix},
   \]
   then $x$ minimizes $\|Ax - b\|_2$.

2. What is the two-norm condition number of the coefficient matrix in part 1 in terms of the singular values of $A$?

3. Give an explicit expression for the inverse of the coefficient matrix, as a block 2-by-2 matrix.

4: Continuous connections  Find the sixth-degree polynomial $p(x)$ that best approximates $\cos(x)$ on $[-\pi, \pi]$ in a least squares sense; that is, minimize

   \[R(p) = \int_{-\pi}^{\pi} (p(x) - \cos(x))^2 \, dx.\]

   Note: For $k$ even, the integrals
   \[b_k = \int_{-\pi}^{\pi} x^k \cos(x) \, dx,
   \]
   satisfy the recurrence $b_0 = 0$ and $b_k = -k \left(2\pi^{k-1} + (k-1)b_{k-2}\right)$. 