HW 5

1: Optimizing on an ellipsoid Suppose M is symmetric and positive definite, so that $x^T Mx = 1$, and let A be a symmetric matrix. Using Lagrange multipliers, show how to minimize $x^T Ax$ subject to $x^T Mx = 1$. Implement this idea as a program

```
function x = p1minimize(A, M)
%
% Returns x s.t. x'*M*x = 1 and x'*A*x is minimal.
```

2: Polynomials from three-term recurrences Consider the polynomials defined by the three-term recurrence

$$(k+1)p_{k+1}(x) - (2k+1)xp_k(x) + kp_{k-1}(x) = 0$$

where $p_0(x) = 1$ and $p_1(x) = x$. By arranging this recurrence into matrix form and scaling the rows and columns, we find that we can write the zeros of $p_n(x)$ as eigenvalues in the problem

$$(T - \lambda I)v = 0$$

where $T \in \mathbb{R}^{n\times n}$ is a symmetric tridiagonal matrix and v consists of scaled values of $p_0(\lambda), \ldots, p_{n-1}(\lambda)$. Write a routine that computes the n roots of $p_n(x)$ by conversion to a symmetric tridiagonal eigenvalue problem:

```
function x = p2roots(n)
```

What relation do the roots of $p_{n-1}(x)$ have to the roots of $p_n(x)$, and why?

Hint: It may be helpful to look at Section 8.5.1 in the text.

3: Eigenvector perturbations Suppose $A = Q\Lambda Q^T$ is a symmetric matrix with all eigenvalues distinct. Let E be another symmetric matrix. For small enough t, we can write a smooth eigenvalue decomposition

$$A + tE = U(t)L(t)U(t)^T$$

where $U(0) = Q$, $L(0) = \Lambda$, and $U(t)$ and $L(t)$ are respectively orthogonal and diagonal for all sufficiently small t. Write a routine to compute $\dot{U}(0)$ and $\dot{L}(0)$. Your routine should have the form

```
function [Udot,Ldot] = p3deriv(Q,Lambda,E)
```

What do you notice about the relationship between closeness of the eigenvalues and the rate at which the eigenvectors change?

Hint: Reduce the problem to the solution of a matrix equation of the form $S\Lambda - \Lambda S^T = G$ where $S = \dot{U}^T U$, and note that S must be skew-symmetric in order for U to remain orthogonal.