PS 8

Due: Wed, May 6

1: Nearest Points For a given point $(a, b) \in \mathbb{R}^2$, we want to find the nearest point (x, y) that lies on the hyperbola $xy = 1$.

1. Write a Lagrangian function $L(x, y, \lambda)$ such that the desired point is a stationary point of L.
2. Write a Newton iteration to find the stationary point of L for $(a, b) = (3, 4)$. Use the starting guess $(x, y, \lambda) = (a, b, 0)$, and demonstrate quadratic convergence.

Note: As the point is to demonstrate a knowledge of Lagrange multipliers, we will not give credit for solutions that eliminate the constraint in advance.

2: Nonlinear Least Squares In this problem, we consider a nonlinear least squares fitting problem in which we fit the coefficient vector β defining a rational function

$$f(x; \beta) = \frac{\beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3}{1 + \beta_5 x + \beta_6 x^2 + \beta_7 x^3}$$

by minimizing

$$\phi(\beta) = \sum_j (f(x_j; \beta) - y_j)^2$$

Your task: Complete the MATLAB script `ps8thuber.m` by filling in the code marked `TODO` with an appropriate solver iteration. You may use Gauss-Newton or Levenberg-Marquardt; I used Gauss-Newton with a line search (necessary to achieve convergence). Terminate when $\|J^T(f - y)\| < 10^{-8}$.

3: Descent directions Suppose that H is symmetric and positive definite, and let $\tilde{p}$ be the solution to the system

$$H\tilde{p} = -\nabla\phi(x) + r$$

where r is a residual vector. If $\kappa(H) = \lambda_{\max}(H)/\lambda_{\min}(H)$, show that if $\kappa(H)\|r\| < \|\nabla\phi\|$ then $\tilde{p}$ is a descent direction.

Hint: Note that if A is any SPD matrix, then we have the two-norm bounds

$$\forall u, v, \quad |u^T A v| \leq \|u\| \|A\| \|v\| = \lambda_{\max}(A) \|u\| \|v\|$$

and

$$\min_{u \neq 0} \frac{u^T A u}{u^T u} = \lambda_{\min}(A), \text{ so } \forall u, u^T A u \geq \lambda_{\min}(A) \|u\|^2.$$