HW 6
Due by CMS (1) and in lecture (2–3) on Wednesday, April 25

Remember that you may (and should!) talk about the problems amongst yourselves, or discuss them with me or the TA, providing attribution for any good ideas you might get – but your final write-up should be your own.

1: Testing Gauss The two-point Gauss-Legendre rule on the interval $[-h, h]$ is

$$\int_{-h}^{h} f(x)\,dx \approx h\left\{f\left(-\frac{h}{\sqrt{3}}\right) + f\left(\frac{h}{\sqrt{3}}\right)\right\}.$$

Based on this rule, implement the following routine:

```
% I = hw6gauss2(f, a, b, n)
%
% Integrate f (passed as a MATLAB function handle) from a to b
% using n panels with two-point Gauss quadrature on each panel.

function I = hw6gauss2(f, a, b, n)
```

In addition, write a testing script to check the following features:

- The interval specified by $[a, b]$ is used. You should test that your code works properly for the case $a = b$, and also for the case where $b < a$ (using the convention that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
- The quadrature rule has degree 3 (i.e. cubics are integrated exactly but quartics need not be).
- The function has the desired order of convergence on the test integrand e^x for the interval $[0, 1]$. You should do this by repeatedly doubling n and showing that the error decreases appropriately.

Your tester should output a diagnostic failure message if `hw6gauss2` is incorrect. I will be checking your test script by making sure that it reports success for a correct implementation of the rule and reports failure for some incorrectly-implemented variants of the rule (including versions that correctly estimate the integral, but do not have the right order of convergence).

2: Simpson's method and square roots On the class web page, you can find the MATLAB function `hw6simpson`, which implements the composite Simpson's rule with n panels. Using this routine, compute

$$I_1 = \int_0^1 \exp(x)\,dx$$
$$I_2 = \int_0^1 \sqrt{x}\,dx$$

for $n = 2^0, 2^1, \ldots, 2^{10}$ panels.

1. On a log-log plot, show $h = 1/n$ against the error in these calculations. What are the slopes of the lines? Why are they not the same?

2. Write ϕ_k for the results of 2^k-panel composite Simpson's rule used to evaluate $\int_0^1 \sqrt{x}\,dx$. Apply the Aitken delta-squared procedure to the sequence ϕ_k (for $k = 0, \ldots, 10$), and to get a new sequence ψ_k (for $k = 0, \ldots, 8$). On a log-log plot, show the error $|\psi_k - I_2|$ as a function of $2^{-(k+2)}$, the smallest mesh size that went into the formation of ψ_k. What is the slope of this log-log plot?

3: What would MATLAB do? On the class web page, you can find the following MATLAB function:

```
% [h, xh] = frecorder(f)
%
% Given a function handle f, create two new function handles:
%   h  - returns the same values as f, and records the inputs
%   xh - returns all the arguments at which h was evaluated

function [h,xh] = frecorder(f)
```

Using this function, it is possible to investigate what MATLAB's quadrature functions do. For example, to see where `quad` samples $\sin(x)$ in order to evaluate $\int_0^1 \sin(x)\,dx$, write:

```
[h, xh] = frecorder(@sin);
result = quad(h, 0, 1);
plot( xh(), sin(xh()), '*' );
fprintf('Number of samples: %d\n', length(xh()));
```

Using this approach, answer the following questions:

1. Consider the following two (equivalent) calculations:

$$I_1 = \int_0^1 x^{1/3} \cos(x) \, dx$$

$$I_2 = \frac{3}{4} + \int_0^1 x^{1/3} (\cos(x) - 1) \, dx$$

   Compare the number of function evaluations required to compute the integral in the first formulation and the integral in the second formulation. Plot where the integrands are sampled in each case, and briefly explain why the points cluster as they do.

2. Describe how you could construct a polynomial $p(x)$ so `quad(p,0,1)` returned zero, but $\int_0^1 p(x) \, dx$ was nonzero.