# Practice prelim 2

The exam is closed book, but you may bring one letter-sized piece of paper with writing on both sides. The actual exam will be two hours.

1. True or false:

   (a) The two norm condition number of $A \in \mathbb{R}^{m \times n}$, $m > n$, is typically defined as $\kappa(A) = \sigma_1/\sigma_n$, where the $\sigma_j$ are the singular values of $A$.

   (b) If $f$ has infinitely many derivatives on $[-1, 1]$ and $p_n(x)$ denotes the polynomial that interpolates $f$ on the uniform mesh $\{x_i = 2i/n - 1\}_{i=0^n}$, then

   $$\max_{x \in [-1,1]} |f(x) - p_n(x)| \to 0 \text{ as } n \to \infty$$

   (c) If $g : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable, then the Newton direction

   $$d = -H_g(x_0)^{-1} \nabla g(x_0)$$

   is a descent direction except at a stationary point.

   (d) If $f$ is a differentiable function, then $\lim_{x_1 \to x_0} f[x_0, x_1] = f'(x_0)$.

   (e) If $x$ minimizes $\|Ax - b\|_2$, then $\|x\|^2 + \|r\|^2 = \|b\|^2$, where $r = Ax - b$.

   (f) If $f(x_*) = 0$ where $f : \mathbb{R}^n \to \mathbb{R}^n$, and if $f$ is continuously differentiable and the Jacobian of $f$ at $x_*$ is nonsingular, then using Newton iteration starting at any point $x_0$ sufficiently close to $x_*$ will converge quadratically.

2. Consider the problem of determining coefficients $\gamma_0, \gamma_1$

   $$\phi(x) = \gamma_0 \exp(\gamma_1 x)$$

   in order to best fit data points $(x_1, \phi_1), (x_2, \phi_2), \ldots, (x_n, \phi_n)$.

   (a) Write a terse MATLAB program to find $\gamma_0, \gamma_1$ to minimize

   $$\sum_{j=1}^{n} \left| \ln \frac{\phi(x_j)}{\phi_j} \right|^2.$$

   Assume the sample points are given in a column vector `x` and the corresponding values are given in a column vector `phi`.

(b) Write MATLAB code for a Newton iteration to minimize

$$\sum_{j=1}^{n} |\phi(x_j) - \phi_j|^2$$

(Don't worry about testing convergence for this exercise – just take ten steps.)

3. Suppose a matrix $A$ satisfies the property

$$\forall i, |a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

Show that for such a matrix, Jacobi iteration converges.
*Hint:* Bound the infinity norm of the error iteration matrix.

4. Write the monomial, Lagrange, and Newton forms of the interpolant through $(-1, 1)$, $(0, 2)$, $(1, 1)$.

5. Write a MATLAB code for the weighted least squares problem

$$\text{minimize } \sum_{j=1}^{n} w_j r_j^2$$

where $r = Ax - b$ and $w_1, \ldots, w_n \geq 0$.