Practice prelim 1

The exam is closed book, but you may bring one letter-sized piece of paper with writing on both sides. The actual exam will be two hours.

1. True or false:

   (a) Suppose we want to solve $Ax = b$, and $\hat{x}$ is an approximate solution. If $\kappa(A)$ is small and $\|A\hat{x} - b\|/\|b\|$ is small, then $\|\hat{x} - x\|/\|x\|$ is small.

   (b) Floating point addition is associative, i.e. `(a+b)+c` and `a+(b+c)` yield the same floating point number in MATLAB.

   (c) IEEE arithmetic obeys the bound $\mathrm{fl}(x \times y) = (x \times y)(1 + \delta)$, $|\delta| < \epsilon_{\mathrm{mach}}$, for every possible floating point number x and y.

   (d) The power basis $\{1, x, \ldots, x^d\}$ is an orthonormal basis for the space of polynomials of degree at most d with the inner product
   $$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

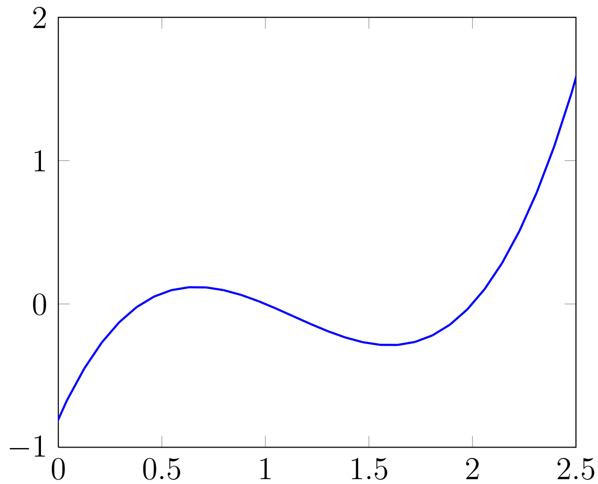
   (e) It is always true that $\|A\|_1 = \|A^T\|_\infty$.

   (f) In double precision floating point (which has 52 bits after the binary point), $\mathrm{fl}(1/3) < 1/3$.

2. Let $f(x) = 1 - \cos(x)$. The function $\hat{f}(x) = \frac{x^2}{2} - \frac{x^4}{24}$ is a truncated Taylor approximation to f near zero. Is this approximation correct to a relative error of 10^{-8} for all x in the range $0 < x < 0.1$?

3. Suppose A is a nonsingular matrix, and we have computed $PA = LU$. Without using A directly or writing explicit inverses (e.g. `inv(L)` or `U^(-1)`), write a MATLAB function that returns the $(1, 1)$ entry of A^{-1}.

4. Suppose the cubic equation $ax^3 + bx^2 + cx + d = 0$ has three positive real roots; for example,



Complete the first line of this MATLAB code to find the smallest root (which should be correct regardless of the values of a, b, c, and d):

```
u = ...;  % Numerically stable computation of an upper bound
x = fzero(@(x) d+x*(c+x*(b+x*a)), [0,u]);
```

5. Suppose f has two continuous derivatives and that f is monotonically increasing and convex ($f'' > 0$ everywhere). Then there is a unique x_* such that $f(x_*) = 0$. Show that for any starting guess x_0, one step of Newton's method applied to f gives $x_1 \geq x_*$.

   Hint: Use Taylor's theorem with remainder about x_0.