HW 7

Due in class on Monday, April 25

Remember that you may (and should!) talk about the problems amongst yourselves, or discuss them with me or the TA, providing attribution for any good ideas you might get – but your final write-up should be your own.

1: Testing Matlab's ODE suite Consider the initial value problem

$$y' = \lambda(y - \sin(t)) + \cos(t), \quad y(0) = 0.$$

The following computational experiments illustrate that even with good solvers like those in MATLAB, innocuous-looking ODEs can cause spectacular problems if used without due caution.

- Verify that $y = \sin(t)$ is the solution independent of λ.
- Use `ode45` and `ode15s` to evaluate the solution for $\lambda = -10^4$ for $0 \leq t \leq 10$. You should turn on statistics reporting, i.e.

  ```
  opt = odeset('Stats', 'on');
  [t,y] = ode45(@hw7p1fun, [0 10], 0, opt);
  ```

  Explain what you observe.
- Use `ode45` and `ode15s` to evaluate the solution for $\lambda = 1$ for $0 \leq t \leq 10$. In each case, plot the difference between the computed solution and the true solution $\sin(t)$. Again, it may be illuminating to turn on statistics reporting. Based on our discussion of step size selection in class, can you guess what went wrong?
- By plotting the error, verify that `ode45` gives an adequate solution if we manually restrict the maximum time step by a modest amount:

  ```
  opt = odeset('Stats', 'on');
  opt = odeset('MaxStep', 0.1);
  [t,y] = ode45(@hw7p1fun, [0 10], 0, opt);
  ```
- What happens if we use `ode15s` with the same max step size of 0.1?

2: Retracing Kepler (Adapted from problem 4.18 of Ascher and Petzold, *Computer Methods for ODEs and DAEs*)

An example *Hamiltonian system* (from orbital mechanics) is a modified Kepler problem in two space dimensions with the potential energy

$$V(q) = -\frac{1}{r} - \frac{\alpha}{2r^3},$$

where $r = \|q\|$ and we will take $\alpha = 0.01$. The differential equations

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\nabla V(q) \end{aligned}$$

conserve a total energy $H(p, q) = \|p\|^2/2 + V(q)$. We will consider the initial value problem with initial conditions:

$$\begin{aligned} q_1(0) &= 1 - \beta & \quad p_1(0) &= 0 \\ q_2(0) &= 0 & \quad p_2(0) &= \sqrt{(1+\beta)/(1-\beta)} \end{aligned}$$

with $\beta = 0.6$.

- Use MATLAB's `ode45` to integrate the problem up to time $T = 100$. Plot the computed orbit, i.e. plot the trajectory of $q(t)$ in the plane; also make a plot of the total energy versus time. What happens when you attempt to integrate up to $T = 500$?

  Note: You may use the function `hw7kepler.m` on CMS.

- Use MATLAB's `ode113` to integrate this problem up to time $T = 500$. Again, plot the computed orbit and total energy versus time.

- Write a routine to solve the equation up to time $T = 500$ with a fixed time step $h = 0.1$ using the *leapfrog method*:

  $$\begin{aligned} q_{k+1/2} &= q_k + \frac{h}{2} p_k \\ p_{k+1} &= p_k - h \nabla V(q_{k+1/2}) \\ q_{k+1} &= q_{k+1/2} + \frac{h}{2} p_{k+1}. \end{aligned}$$

  Again, plot the computed orbit and total energy versus time.