Localizing Nonlinear Eigenvalues: Theory and Applications

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Vibrations are everywhere, and so too are the eigenvalues associated with them. As mathematical models invade more and more disciplines, we can anticipate a demand for eigenvalue calculations in an ever richer variety of contexts.

– Beresford Parlett, The Symmetric Eigenvalue Problem
Why Eigenvalues?

As a student in a first ODE course:

\[ y' - Ay = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda I - A)v = 0. \]

Me: “How do I compute this?”

\[ p(\lambda) = \det(\lambda I - A) = 0. \]
Why Nonlinear Eigenvalues?

\[ y' - Ay = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad (\lambda I - A)v = 0 \]

\[ y'' + By' + Ky = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad (\lambda^2 I + \lambda B + K)v = 0 \]

\[ y' - Ay - By(t - 1) = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad (\lambda I - A - Be^{-\lambda})v = 0 \]

\[ T(d/dt)y = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad T(\lambda)v = 0 \]

- Higher-order ODEs
- Dynamic element formulations
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
My motivation

\[ T(\omega)v \equiv (K - \omega^2 M + G(\omega))v = 0 \]

**Wanted:** Perturbation theory justifying a terrible estimate of \( G(\omega) \)
Nonlinear eigenvalue problem

\[ T(\lambda)v = 0, \quad v \neq 0. \]

where

- \( T : \Omega \to \mathbb{C}^{n \times n} \) analytic, \( \Omega \subset \mathbb{C} \) simply connected
- Regularity: \( \det(T) \neq 0 \)

Nonlinear spectrum: \( \Lambda(T) = \{ z \in \Omega : T(z) \text{ singular} \} \).

**Goal:** Use analyticity to *compare* and to *count*
Analytic \( f, g : \Omega \to \mathbb{C} \)

Winding # \( \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz \)

Theorem

Rouché (1862): \( |g| < |f| \) on \( \Gamma \) \( \implies \) same # zeros of \( f, f + g \)

Gohberg-Sigal (1971): \( \|T^{-1}E\| < 1 \) on \( \Gamma \) \( \implies \) same # eigs of \( T, T + E \)
Comparing NEPs

Suppose

$$T, E : \Omega \to \mathbb{C}^{n \times n} \text{ analytic}$$

$$\Gamma \subset \Omega \text{ a simple closed contour}$$

$$T(z) + sE(z) \text{ nonsingular } \forall s \in [0, 1], z \in \Gamma$$

Then $T$ and $T + E$ have the same number of eigenvalues inside $\Gamma$.

**Pf:** Constant winding number around $\Gamma$. 
Nonlinear pseudospectra

\[ \Lambda_\epsilon(T) \equiv \{ z \in \Omega : \| T(z)^{-1} \| > \epsilon^{-1} \} \]
$E$ analytic, $\|E(z)\| < \epsilon$ on $\Omega_\epsilon$. Then

$$\Lambda(T + E) \cap \Omega_\epsilon \subset \Lambda_\epsilon(T) \cap \Omega_\epsilon$$

Also, if $\mathcal{U}_\epsilon$ a component of $\Lambda_\epsilon$ and $\overline{\mathcal{U}}_\epsilon \subset \Omega_\epsilon$, then

$$|\Lambda(T + E) \cap \mathcal{U}_\epsilon| = |\Lambda(T) \cap \mathcal{U}_\epsilon|$$
Pseudospectral comparison

- Most useful when $T$ is linear
- Even then, can be expensive to compute!
- What about related tools?
The Gershgorin picture (linear case)

\[ A = D + F, \quad D = \text{diag}(d_i), \quad \rho_i = \sum_j |f_{ij}| \]
Gershgorin $(+\epsilon)$

Write $A = D + F$, $D = \text{diag}(d_1, \ldots, d_n)$. Gershgorin disks are:

$$G_i = \left\{ z \in \mathbb{C} : |z - d_i| \leq \sum_j |f_{ij}| \right\}.$$

Useful facts:

- Spectrum of $A$ lies in $\bigcup_{i=1}^m G_i$
- $\bigcup_{i \in \mathcal{I}} G_i$ disjoint from other disks $\implies$ contains $|\mathcal{I}|$ eigenvalues.

Pf:

$A - zI$ strictly diagonally dominant outside $\bigcup_{i=1}^m G_i$. Eigenvalues of $D - sF$, $0 \leq s \leq 1$, are continuous.
Nonlinear Gershgorin

Write $T(z) = D(z) + F(z)$. Gershgorin regions are

$$G_i = \left\{ z \in \mathbb{C} : |d_i(z)| \leq \sum_j |f_{ij}(z)| \right\}.$$

Useful facts:

- Spectrum of $T$ lies in $\bigcup_{i=1}^m G_i$
- Bdd connected component of $\bigcup_{i=1}^m G_i$ strictly in $\Omega$
  $\implies$ same number of eigs of $D$ and $T$ in component
  $\implies$ at least one eig per component of $G_i$ involved

**Pf:** Strict diag dominance test + continuity of eigs
Example I: Hodeler

\[ T(z) = (e^z - 1)B + z^2 A - \alpha I, \quad A, B \in \mathbb{R}^{8 \times 8} \]
Comparison to simplified problem

Bauer-Fike idea: apply a similarity!

\[ T(z) = (e^z - 1)B + z^2 A - \alpha I \]

\[ \tilde{T}(z) = U^T T(z) U \]
\[ = (e^z - 1)D_B + z^2 I - \alpha E \]
\[ = D(z) - \alpha E \]

\[ G_i = \{ z : |\beta_i (e^z - 1) + z^2| < \rho_i \}. \]
Gershgorin regions
A different comparison

Approximate \( e^z - 1 \) by a Chebyshev interpolant:

\[
T(z) = (e^z - 1)B + z^2A - \alpha I \\
\tilde{T}(z) = q(z)B + z^2A - \alpha
\]

\[
T(z) = \tilde{T}(z) + r(z)B
\]

Linearize \( \tilde{T} \) and transform both:

\[
\tilde{T}(z) \mapsto D_C - zI \\
T(z) \mapsto D_C - zI + r(z)E
\]

Restrict to \( \Omega_\epsilon = \{ z : |r(z)| < \epsilon \} \):

\[
G_i \subset \hat{G}_i = \{ z : |z - \mu_i| < \rho_i \epsilon \}, \quad \rho_i = \sum_j |e_{ij}|
\]
Spectrum of $\tilde{T}$
\( \hat{G}_i \) for \( \epsilon < 10^{-10} \)
\( \hat{G}_i \) for \( \epsilon = 0.1 \)
$\hat{G}_i$ for $\epsilon = 1.6$
Example II: Resonance problem

\[ \psi(0) = 0 \]

\[ \left( -\frac{d^2}{dx^2} + V - \lambda \right) \psi = 0 \quad \text{on} \ (0, b), \]

\[ \psi'(b) = i\sqrt{\lambda} \psi(b), \]
Reduction via shooting

\[ \psi(0) = 0, \]

\[ R_{0a}(\lambda) \begin{bmatrix} \psi(0) \\ \psi'(0) \end{bmatrix} = \begin{bmatrix} \psi(a) \\ \psi'(a) \end{bmatrix}, \]

\[ R_{ab}(\lambda) \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix} = \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix}, \]

\[ \psi'(b) = i\sqrt{\lambda}\psi(b) \]
Reduction via shooting

First-order form:
\[ \frac{du}{dx} = \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix} u, \ \text{where } u(x) \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}. \]

On region \((c, d)\) where \(V\) is constant:
\[ u(d) = R_{cd}(\lambda)u(c), \quad R_{cd}(\lambda) = \exp\left( (d - c) \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix} \right) \]

Reduce resonance problem to 6D NEP:
\[ T(\lambda)u_{\text{all}} \equiv \begin{bmatrix} R_{0a}(\lambda) & -I & 0 \\ 0 & R_{ab}(\lambda) & -I \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(a) \\ u(b) \end{bmatrix} = 0. \]
Expansion via rational approximation

\[ u(0) \begin{array}{cccc}
 u(a) \\
 u(b) \\
 \vdots \\
 \end{array} \times = 0 \]
Analyzing the expanded system

- $\hat{T}(z)$ is a Schur complement in $K - zM$
  - So $\Lambda(\hat{T})$ is easy to compute.
- Or: think $T(z)$ is a Schur complement in $K - zM + E(z)$
- Compare $\hat{T}(z)$ to $T(z)$ or compare $K - zM + E(z)$ to $K - zM$
Analyzing the expanded system

Q: Can we find all eigs in a region \textit{not missing anything}?

Concrete plan ($\epsilon = 10^{-8}$)

- $T =$ shooting system
- $\hat{T} =$ rational approximation
- Find region $D$ with boundary $\Gamma$ s.t.
  - $D \subset \Omega_\epsilon$ (i.e. $\|T - \hat{T}\| < \epsilon$ on $D$)
  - $\Gamma$ does not intersect $\Lambda_\epsilon(T)$

$\implies$ Same eigenvalue counts for $T$, $\hat{T}$

$\implies$ Eigs of $\hat{T}$ in components of $\Lambda_\epsilon(T)$
  - Converse holds if $D \subset \Omega_{\epsilon/2}$

Can refine eigs of $\hat{T}$ in $D$ via Newton.
Resonance approximation
Resonance approximation
Localization theorems for nonlinear eigenvalues.
David Bindel and Amanda Hood, SIMAX 34(4), 2013

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