Music of the Microspheres

Eigenvalue problems from micro-gyro design

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I. Schur (1875–1941)

Prelude

Suppose D diagonal and nonsingular and

$$D^{-1}AD = A$$

Then $a_{ij} = a_{ij}d_i/d_j$. So $d_i \neq d_j \implies a_{ij} = 0$.

Suppose Q orthogonal and

$$Q^*AQ = A.$$

Then each max invariant subspace of Q is invariant for A.

Prelude

A real symmetric, $Q^*AQ = A$, Q has non-real eigenvalues

$$\Longrightarrow$$

A has degenerate eigenvalues

Example:

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Role of Symmetry

General picture:

- Matrix (or operator) A commutes with action of some group.
- Invariant subspaces of A are representations.
- Canonical subspaces for different irreducible representations.
- Degeneracy for non-abelian groups
  (or cycles longer than two in real symmetric case).

The Real Beginning: A Perfect Wine Glass

Free Vibrations

Generalized eigenvalue problem in weak form

$$\forall \mathbf{w} \in \mathcal{V}, \quad -\omega^2 b(\mathbf{w}, \mathbf{u}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

where

$$b(\mathbf{w}, \mathbf{u}) = \int_{\mathcal{B}} \rho \mathbf{w} \cdot \mathbf{u} \, d\mathcal{B},$$

$$a(\mathbf{w}, \mathbf{u}) = \int_{\mathcal{B}} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{C} : \boldsymbol{\varepsilon}(\mathbf{u}) \, d\mathcal{B}.$$

Role of Symmetry

Generalized eigenvalue problem in weak form

$$\forall \mathbf{w} \in \mathcal{V}, \quad -\omega^2 b(\mathbf{w}, \mathbf{u}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

and suppose for $Q : \mathcal{V} \to \mathcal{V}$,

$$a(Q\mathbf{w}, Q\mathbf{u}) = a(\mathbf{w}, \mathbf{u})$$
$$b(Q\mathbf{w}, Q\mathbf{u}) = b(\mathbf{w}, \mathbf{u}).$$

Then a max invariant subspace of Q is invariant for the GEP.

Specific picture:

- Rotational symmetry $\implies$ decomposition by Fourier analysis.
- Non-axisymmetric modes come in degenerate pairs (sine/cosine)

Fourier Expansion and Axisymmetric Shapes

Decompose into symmetric $\mathbf{u}^c$ and antisymmetric $\mathbf{u}^s$ in y:

$$\mathbf{u}^c = \sum_{m=0}^{\infty} \mathbf{\Phi}_m^c(\theta)\mathbf{u}_m^c(r,z), \qquad \mathbf{u}^s = \sum_{m=0}^{\infty} \mathbf{\Phi}_m^s(\theta)\mathbf{u}_m^s(r,z)$$

where

$$\begin{aligned}
\mathbf{\Phi}_m^c(\theta) &= \operatorname{diag}\left(\cos(m\theta),\ \sin(m\theta),\ \cos(m\theta)\right) \\
\mathbf{\Phi}_m^s(\theta) &= \operatorname{diag}\left(-\sin(m\theta),\ \cos(m\theta), -\sin(m\theta)\right).
\end{aligned}$$

Modes involve only one azimuthal number m; degenerate for $m > 1$.

Preserve structure in FE: shape functions $N_j(r,z)\mathbf{\Phi}_m^{c,s}(\theta)$

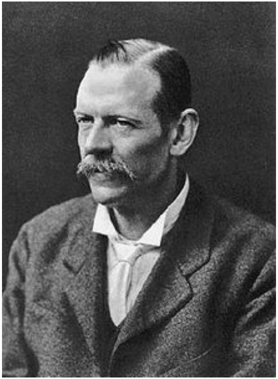
Mass and Stiffness Structure

Finite element system: $\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{K}\mathbf{u}^h = 0$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0^{cc} & & & & & & & \\ & \mathbf{K}_1^{ss} & & & & & & \\ & & \mathbf{K}_1^{cc} & & & & & \\ & & & \mathbf{K}_2^{ss} & & & & \\ & & & & \mathbf{K}_2^{cc} & & & \\ & & & & & \ddots & & \\ & & & & & & \mathbf{K}_M^{ss} & \\ & & & & & & & \mathbf{K}_M^{cc} \end{bmatrix}$$

Mass has same structure.

G. H. Bryan (1864–1928)



- Fellow of the Royal Society (1895)
- *Stability in Aviation* (1911)
- Thermodynamics, hydrodynamics

Bryan was a friendly, kindly, very eccentric individual...

*(*Obituary Notices of the FRS*)*

... if he sometimes seemed a colossal buffoon, he himself did not help matters by proclaiming that he did his best work under the influence of alcohol.

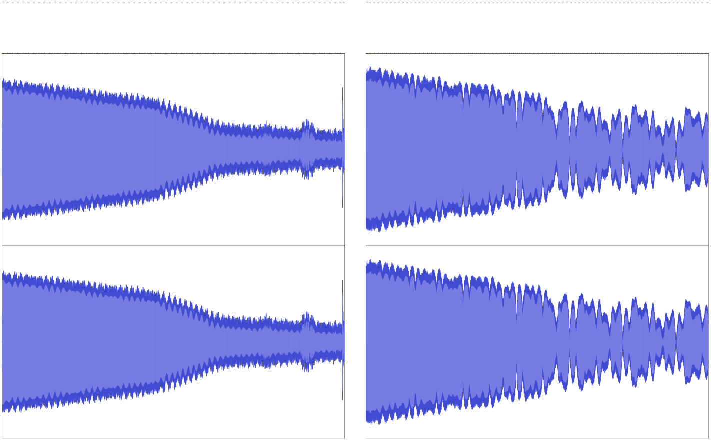
(Williams, J.G., The University College of North Wales, 1884–1927*)*

Bryan's Experiment

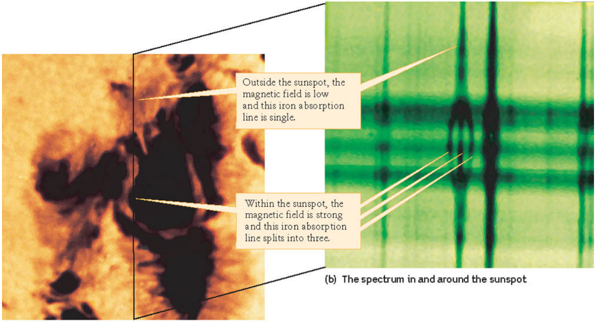
"On the beats in the vibrations of a revolving cylinder or bell"
by G. H. Bryan, 1890

Bryan's Experiment Today

The Beat Goes On



Bryan, Zeeman, Stark, ...



(a) A sunspot

(b) The spectrum in and around the sunspot

This is a common picture:

- Symmetry leads to degenerate modes
- Perturbations split (some) degeneracies

Fictitious Forces

Free vibration problem in weak form

$$\forall \mathbf{w} \in \mathcal{V}, \quad b(\mathbf{w}, \mathbf{a}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

where b and a are the stiffness and mass forms and

$$\mathbf{a} = \ddot{\mathbf{u}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{u}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) + \dot{\boldsymbol{\Omega}} \times \mathbf{x}$$

Assumption: Ω^2/ω^2 and $\dot{\boldsymbol{\Omega}}/\omega^2$ are negligible.

The Eigenvalue Problem

Start from weak form (discarding centripetal + Euler forces):

$$\forall \mathbf{w} \in \mathcal{V}, \quad b(\mathbf{w}, \ddot{\mathbf{u}}) + 2b(\mathbf{w}, \boldsymbol{\Omega} \times \dot{\mathbf{u}}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

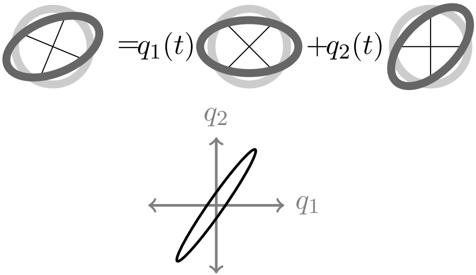
Assume $\mathbf{u}$ and $\mathbf{w}$ in an appropriate FE space:

$$\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{C}\dot{\mathbf{u}}^h + \mathbf{K}\mathbf{u} = 0$$

where $\mathbf{C}$ is skew and “small.”

Bryan’s effect = symmetry breaking by gyroscopic perturbation.

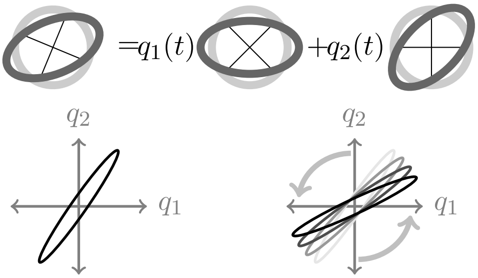
Reduced Dynamics: Stationary Frame



Consider free vibration consisting of two modes of oscillation:

$$\ddot{\mathbf{q}} + \omega_0^2 \mathbf{q} = 0, \quad \mathbf{q}(t) \in \mathbb{R}^2.$$

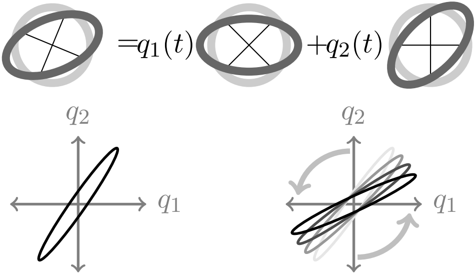
Reduced Dynamics: Rotating Frame



Rate of rotation is $\Omega \ll \omega_0$:

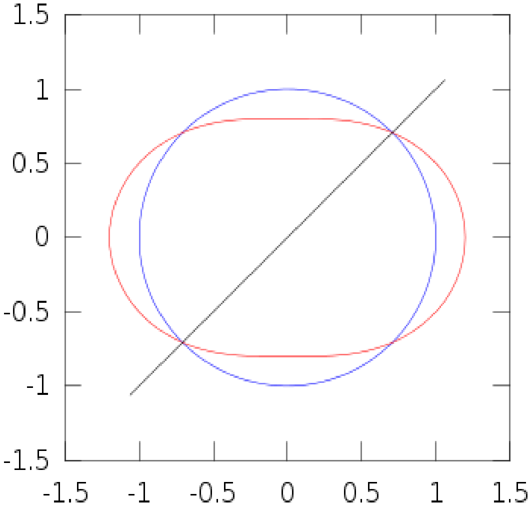
$$\ddot{\mathbf{q}} + 2\beta\Omega\mathbf{J}\dot{\mathbf{q}} + \omega_0^2\mathbf{q} = 0, \qquad \mathbf{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A General Picture: Rotating Frame



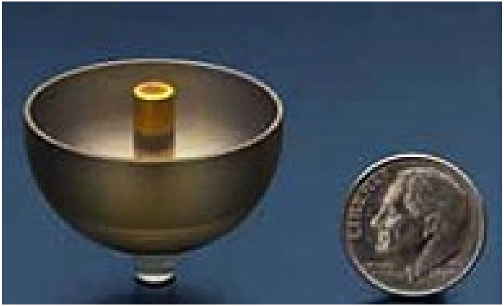
$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \approx \begin{bmatrix} \cos(-\beta\Omega t) & -\sin(-\beta\Omega t) \\ \sin(-\beta\Omega t) & \cos(-\beta\Omega t) \end{bmatrix} \begin{bmatrix} q_1^0(t) \\ q_2^0(t) \end{bmatrix}.$$

Measuring Rotation



Foucault in Solid State

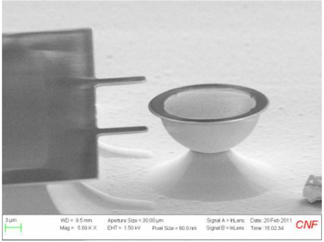
A Small Application

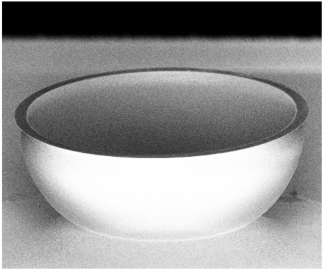


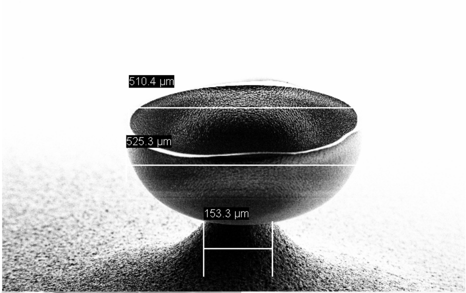
Northrup-Grumman HRG
(developed c. 1965–early 1990s)

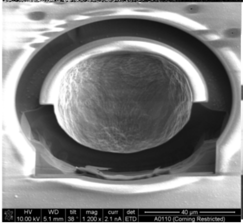
See: Donald Mackenzie. *Knowing Machines: Essays on Technical Change*.

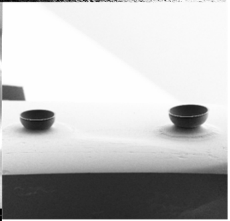
A Smaller Application (Cornell)



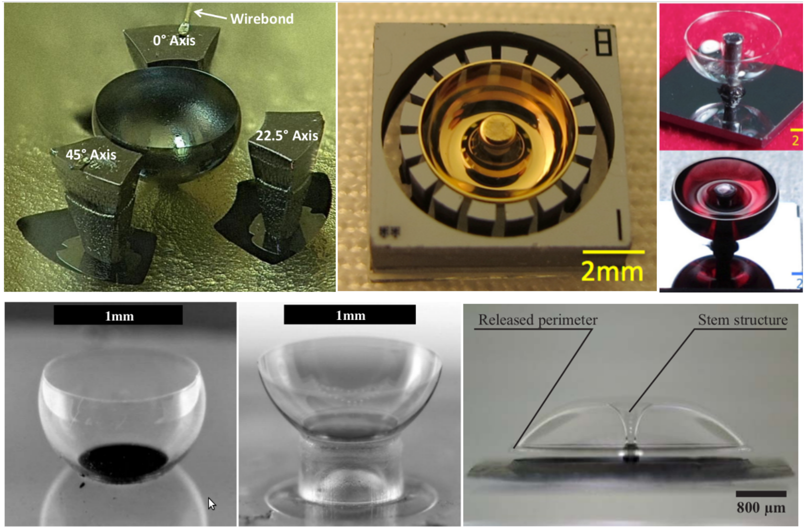




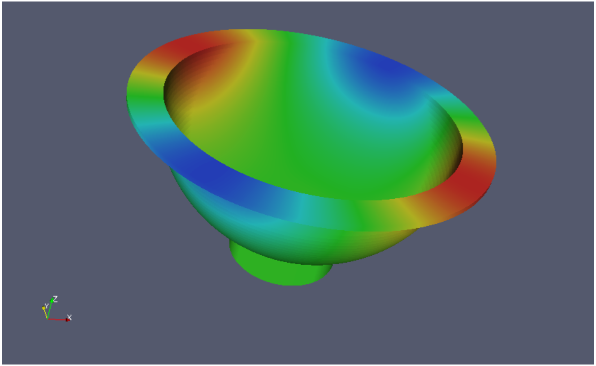




A Smaller Application (UMich, GA Tech, Irvine)



The Perturbation Picture



Perturbations split degenerate modes:

- Coriolis forces (good)
- Imperfect fab (bad, but physical)
- Discretization error (non-physical)

Finite Elements

Start from weak form (discarding centripetal + Euler forces):

$$\forall \mathbf{w}, \quad b(\mathbf{w}, \ddot{\mathbf{u}}) + 2b(\mathbf{w}, \boldsymbol{\Omega} \times \dot{\mathbf{u}}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

Assume $\mathbf{u}$ and $\mathbf{w}$ in an appropriate FE space:

$$\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{C}\dot{\mathbf{u}}^h + \mathbf{K}\mathbf{u} = 0$$

Naive FE tool with auto-mesher $\implies$ numerical symmetry breaking.
Trig shape functions to block diagonalize $\mathbf{K}$ and $\mathbf{M}$.

What of $\mathbf{C}$?

Imperfections: Cross-Axis Sensitivity

Discretize $2b(\mathbf{w}, \mathbf{\Omega} \times \dot{\mathbf{u}})$:

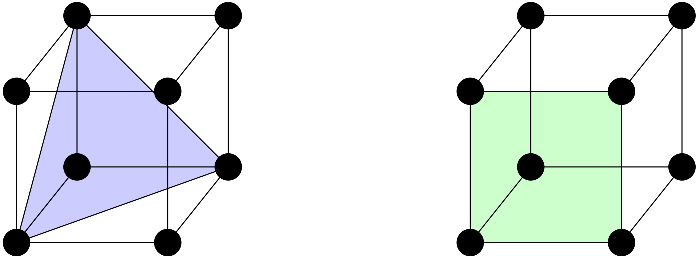
$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} & & & \\ \mathbf{C}_{10} & \mathbf{C}_{11} & \mathbf{C}_{12} & & \\ & \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Off-diagonal blocks come from cross-axis sensitivity:

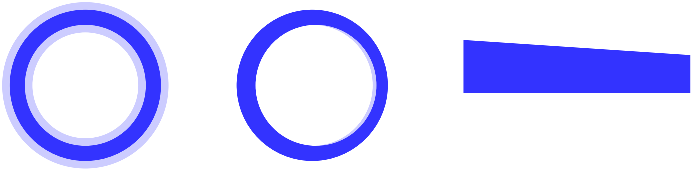
$$\mathbf{\Omega} = \Omega_z \mathbf{e}_z + \mathbf{\Omega}_{r\theta} = \begin{bmatrix} 0 \\ 0 \\ \Omega_z \end{bmatrix} + \begin{bmatrix} \cos(\theta)\Omega_x - \sin(\theta)\Omega_y \\ \sin(\theta)\Omega_x + \cos(\theta)\Omega_y \\ 0 \end{bmatrix}.$$

Neglect cross-axis effects ($O(\Omega^2/\omega_0^2)$, like centrifugal effect).

Imperfections: Etch Anisotropy



Imperfections: Processing Effects



Representing the Perturbation

Map axisymmetric $\mathcal{B}_0 \to$ real $\mathcal{B}$:

$$\mathbf{R} \in \mathcal{B}_0 \quad \mapsto \quad \mathbf{r} = \mathbf{R} + \epsilon \boldsymbol{\psi}(\mathbf{R}) \in \mathcal{B}.$$

Write weak form in $\mathcal{B}_0$ geometry:

$$b(\mathbf{w}, \mathbf{a}) = \int_{\mathcal{B}_0} \rho \mathbf{w} \cdot \mathbf{a} \, J \, d\mathcal{B}_0,$$

$$a(\mathbf{w}, \mathbf{u}) = \int_{\mathcal{B}_0} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{C} : \boldsymbol{\varepsilon}(\mathbf{u}) \, J \, d\mathcal{B}_0,$$

where $J = \det(\mathbf{I} + \epsilon \mathbf{F})$ and $\boldsymbol{\varepsilon}(\mathbf{w}) = (\mathbf{F}^{-T} \nabla \mathbf{u})^s$ with $\mathbf{F} = \partial \boldsymbol{\psi} / \partial \mathbf{R}$.

Decomposing ψ

Do Fourier decomposition of ψ, too! Consider case where

$$\begin{aligned} m &= \text{only azimuthal number of } \mathbf{w} \\ n &= \text{only azimuthal number of } \mathbf{u} \\ p &= \text{only azimuthal number of } \psi \end{aligned}$$

Dominant Fourier modes:

- Over/under etch ($p = 0$)
- Mask misalignment ($p = 1$)
- Thickness variations ($p = 1$)
- Anisotropy of etching single-crystal Si ($p = 3$ or $p = 4$)

Selection Rules

Consider

$$a(\mathbf{w},\mathbf{u}) = \int_{\mathcal{B}_0} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{C} : \boldsymbol{\varepsilon}(\mathbf{u})\, J\, d\mathcal{B}_0$$

Dependence of terms on θ is:

$$\begin{array}{lll} \boldsymbol{\varepsilon}(\mathbf{w})\text{:} & t(m\theta) & \big(O(1) + O(\epsilon)t(p\theta) + O(\epsilon^2)t(2p\theta) + \dots\big) \\ \boldsymbol{\varepsilon}(\mathbf{w})\text{:} & t(n\theta) & \big(O(1) + O(\epsilon)t(p\theta) + O(\epsilon^2)t(2p\theta) + \dots\big) \\ \mathbf{J}\text{:} & & \big(O(1) + O(\epsilon)t(p\theta) + O(\epsilon^2)t(2p\theta) + \dots\big) \end{array}$$

where $t(n\theta) =$ "trig in $n\theta$"

Then we have *selection rules*

$$a(\mathbf{w},\mathbf{u}) = \begin{cases} O(\epsilon^k), & |m \pm n| = kp \\ 0, & \text{otherwise} \end{cases}$$

Similar picture for b.

Block Matrix Structure

Ex: $p = 2$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0 & & \epsilon & & \epsilon^2 & & \epsilon^3 & \\ & \mathbf{K}_1 & & \epsilon & & \epsilon^2 & & \\ \epsilon & & \mathbf{K}_2 & & \epsilon & & \epsilon^2 & \\ & \epsilon & & \mathbf{K}_3 & & \epsilon & & \\ \epsilon^2 & & \epsilon & & \mathbf{K}_4 & & \epsilon & \\ & \epsilon^2 & & \epsilon & & \mathbf{K}_5 & & \\ \epsilon^3 & & \epsilon^2 & & \epsilon & & \mathbf{K}_6 & \\ & & & & & & & \ddots \end{bmatrix}$$

Note: Diagonal blocks perturbed at $O(\epsilon^k)$ when $kp = 2m$.
$\implies$ eigenvalues unchanged at first order for $p \neq 2m$.

Block Matrix Structure

Let $\mathbf{T}(\lambda) = \mathbf{K} - \lambda\mathbf{M}$, solve $\mathbf{T}(\lambda)v = 0$.

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_0 & & \epsilon & & \epsilon^2 & & \epsilon^3 & \\ & \mathbf{T}_1 & & \epsilon & & \epsilon^2 & & \\ \epsilon & & \mathbf{T}_2 & & \epsilon & & \epsilon^2 & \\ & \epsilon & & \mathbf{T}_3 & & \epsilon & & \\ \epsilon^2 & & \epsilon & & \mathbf{T}_4 & & \epsilon & \\ & \epsilon^2 & & \epsilon & & \mathbf{T}_5 & & \\ \epsilon^3 & & \epsilon^2 & & \epsilon & & \mathbf{T}_6 & \\ & & & & & & & \ddots \end{bmatrix}$$

Schur complement onto $\mathbf{T}_2$ block. No effects at $O(\epsilon)$...

Block Matrix Structure

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Schur complement onto $\mathbf{T}_2$ block. At $O(\epsilon^2)$, see coupling to two blocks.

Impact of Selection Rules

- Fast FEA: Can neglect some wave numbers / blocks
- All assuming no accidental (near) degeneracies
- First order: Only need diagonal blocks
- Second order: Keep diagonal plus “directly coupled”.

Qualitative Information

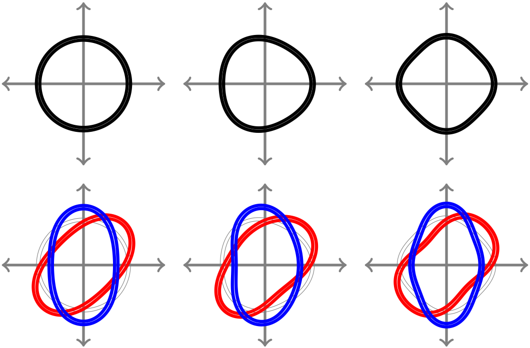
Operating wave number m, perturbation number p:

$p = 2m$	frequencies split by $O(\epsilon)$
$kp = 2m$	frequencies split at most $O(\epsilon^2)$
$p \not| 2m$	frequencies change at $O(\epsilon^2)$, *no* split
$p = 1$ $p = 2m \pm 1$	$O(\epsilon)$ cross-axis coupling.

Note:

- $m = 2$ affected at first order by $p = 0$ and $p = 4$ (and $O(\epsilon^2)$) split from $p = 1$ and $p = 2$).
- $m = 3$ affected at first order by $p = 0$ and $p = 6$ (and $O(\epsilon^2)$) split from $p = 1$ and $p = 3$).

Analyzing Imperfect Rings



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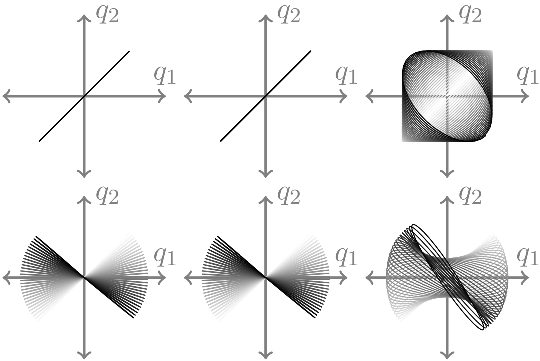
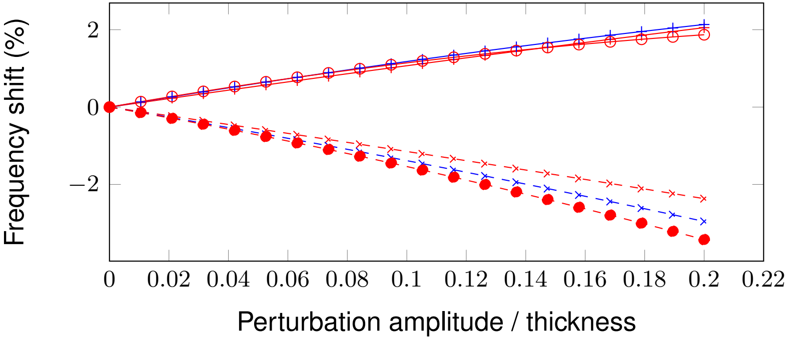


Mode Split for Rings: $\psi(r, \theta) = (\cos(2m\theta), 0)$.



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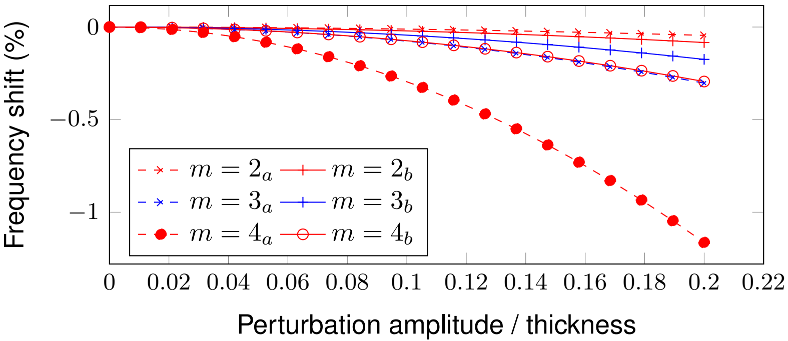
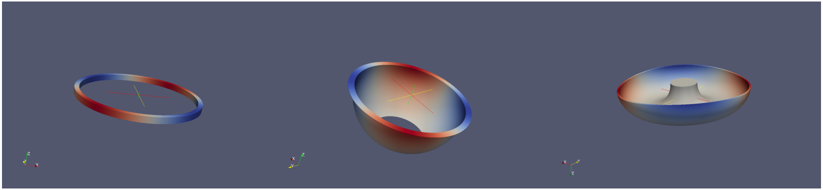


Beyond Rings: AxFEM



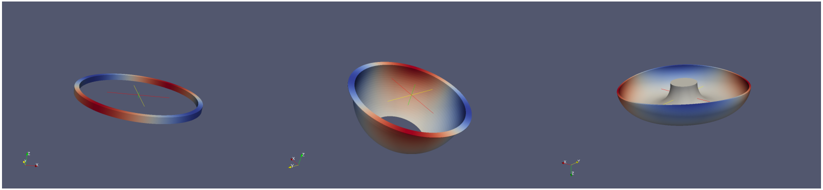
- Mapped finite / spectral element formulation
- Low-order polynomials through thickness
- High-order polynomials along length
- Trig polynomials in θ
- Agrees with results reported in literature
- Computes sensitivity to geometry, material parameters, etc.

Further Steps

Lots of possible directions:

- Symmetry breaking through damping?
- Integration with fabrication simulation?
- Joint optimization of geometry and fabrication?

Thank You



Yilmaz and Bindel
“Effects of Imperfections on Solid-Wave Gyroscope Dynamics”
Proceedings of IEEE Sensors 2013, Nov 3–6.

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