A Tale of Two Eigenvalue Problems

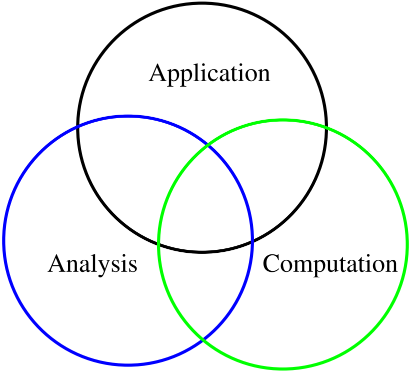
Music of the Microspheres + Hearing the Shape of a Graph

David Bindel

Department of Computer Science
Cornell University

Brown Bag Talk, 26 Nov 2013

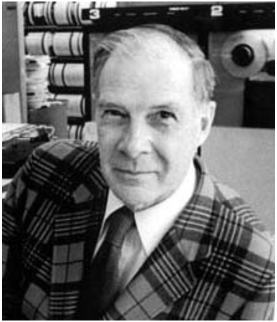
The Computational Science & Engineering Picture



- MEMS
- Smart grids
- Networks

- Linear algebra
- Approximation theory
- Symmetry + structure

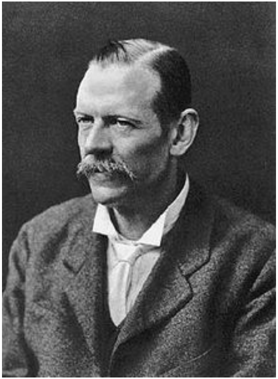
- HPC / cloud
- Simulators
- Little languages



The purpose of computing is insight, not numbers.

Richard Hamming

G. H. Bryan (1864–1928)



- Fellow of the Royal Society (1895)
- *Stability in Aviation* (1911)
- Thermodynamics, hydrodynamics

Bryan was a friendly, kindly, very eccentric individual...

*(*Obituary Notices of the FRS*)*

... if he sometimes seemed a colossal buffoon, he himself did not help matters by proclaiming that he did his best work under the influence of alcohol.

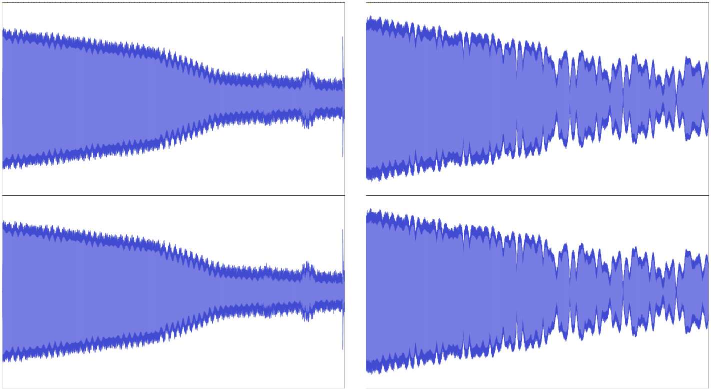
(Williams, J.G., The University College of North Wales, 1884–1927*)*

Bryan's Experiment

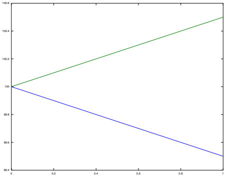
"On the beats in the vibrations of a revolving cylinder or bell"
by G. H. Bryan, 1890

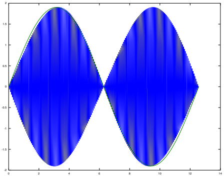
Bryan's Experiment Today

The Beat Goes On



The Beat Goes On





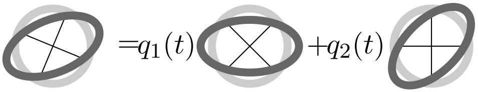
Free vibrations in a rotating frame (simplified):

$$\ddot{\mathbf{q}} + 2\beta\Omega\mathbf{J}\dot{\mathbf{q}} + \omega_0^2\mathbf{q} = 0, \qquad \mathbf{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalue problem: $\left(-\omega^2\mathbf{I} + 2i\omega\beta\Omega\mathbf{J} + \omega_0^2\right) q = 0.$

Solutions: $\omega \approx \Omega_0 \pm \beta\Omega$. $\implies$ beating $\propto \Omega$!

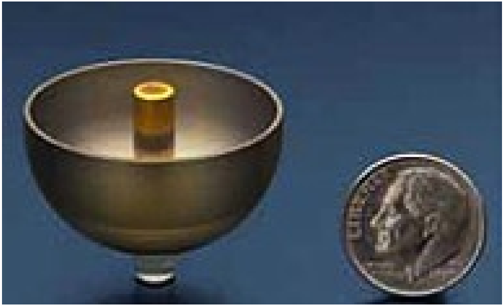
A General Picture



$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \approx \begin{bmatrix} \cos(-\beta\Omega t) & -\sin(-\beta\Omega t) \\ \sin(-\beta\Omega t) & \cos(-\beta\Omega t) \end{bmatrix} \begin{bmatrix} q_1^0(t) \\ q_2^0(t) \end{bmatrix}.$$

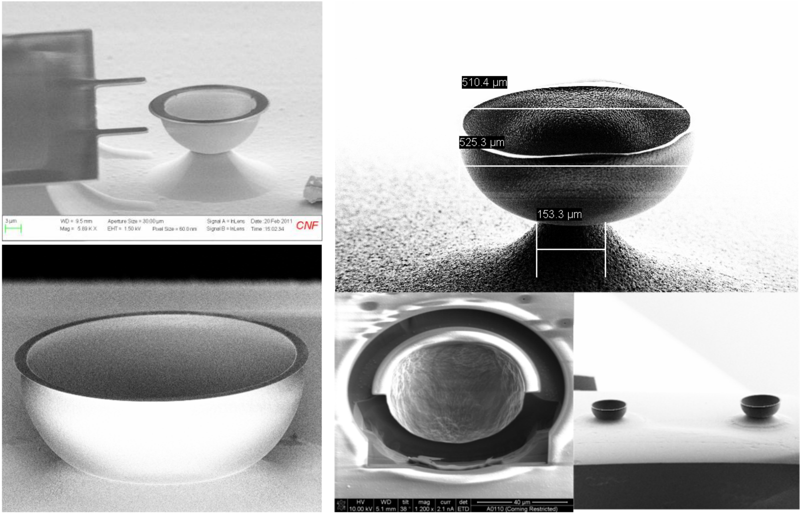
Foucault in Solid State

A Small Application

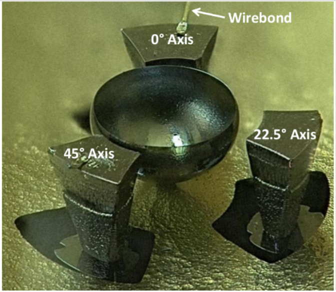


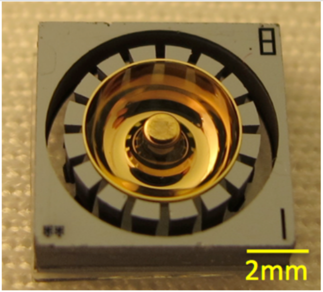
Northrup-Grummond HRG
(developed c. 1965–early 1990s)

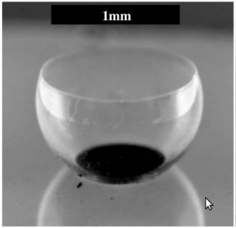
A Smaller Application (Cornell)

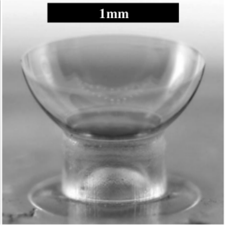


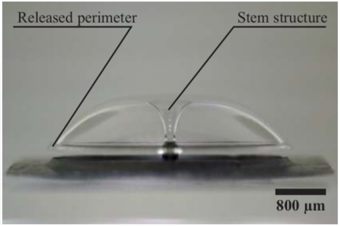
A Smaller Application (UMich, GA Tech, Irvine)



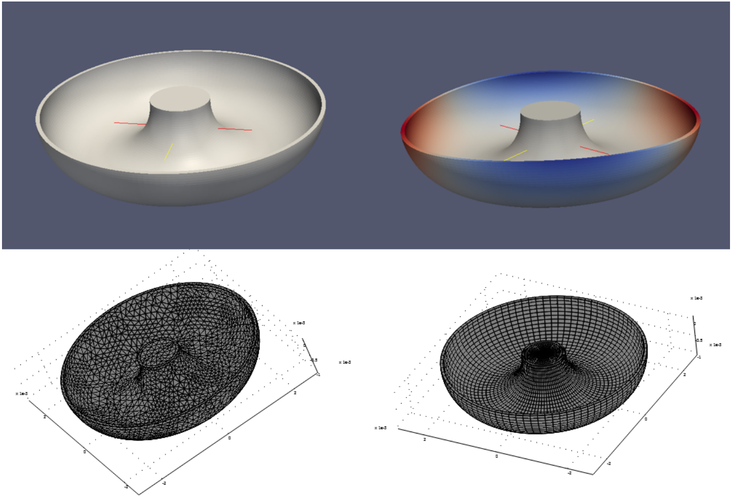




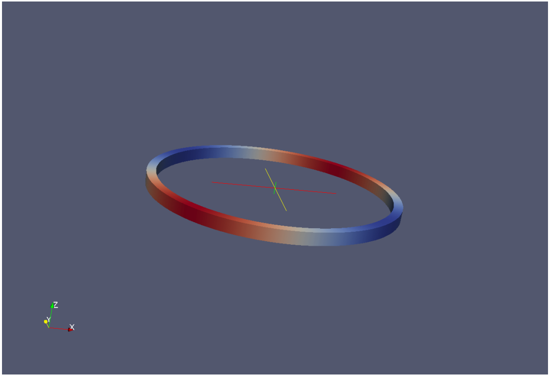




Uncritical FEA: Fail!



The Perturbation Picture



Perturbations split degenerate modes:

- Coriolis forces (good)
- Imperfect fab (bad, but physical)
- Discretization error (non-physical)

Perfect Geometry

Free vibration problem in weak form

$$\forall \mathbf{w}, \quad b(\mathbf{w}, \mathbf{a}) + a(\mathbf{w}, \mathbf{u}) = 0.$$

where

$$\mathbf{a} = \ddot{\mathbf{u}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{u}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) + \dot{\boldsymbol{\Omega}} \times \mathbf{x}$$

Expand $\mathbf{u}$ (and $\mathbf{w}$) in Fourier series:

$$\mathbf{u}(r, z, \theta) = \sum_{m=0}^{\infty} \boldsymbol{\Phi}_m^c(\theta)\mathbf{u}_m^c(r, z) + \sum_{m=0}^{\infty} \boldsymbol{\Phi}_m^s(\theta)\mathbf{u}_m^s(r, z)$$

Discretize by finite elements:

$$\mathbf{M}\ddot{\mathbf{u}}^h + \mathbf{C}\dot{\mathbf{u}}^h + \mathbf{K}\ddot{\mathbf{u}}^h = 0$$

where $\mathbf{C}$ comes from Coriolis term ($2b(\mathbf{w}, \boldsymbol{\Omega} \times \dot{\mathbf{u}})$).

Block Diagonal Structure

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0^{cc} & & & & & & & \\ & \mathbf{K}_1^{ss} & & & & & & \\ & & \mathbf{K}_1^{cc} & & & & & \\ & & & \mathbf{K}_2^{ss} & & & & \\ & & & & \mathbf{K}_2^{cc} & & & \\ & & & & & \ddots & & \\ & & & & & & \mathbf{K}_M^{ss} & \\ & & & & & & & \mathbf{K}_M^{cc} \end{bmatrix}$$

Mass has same structure.

Block Structure of Finite Element Matrix

Discretize $2b(\mathbf{w}, \mathbf{\Omega} \times \dot{\mathbf{u}})$:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{01} & & & \\ \mathbf{C}_{10} & \mathbf{C}_{11} & \mathbf{C}_{12} & & \\ & \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Off-diagonal blocks come from cross-axis sensitivity:

$$\mathbf{\Omega} = \Omega_z \mathbf{e}_z + \mathbf{\Omega}_{r\theta} = \begin{bmatrix} 0 \\ 0 \\ \Omega_z \end{bmatrix} + \begin{bmatrix} \cos(\theta)\Omega_x - \sin(\theta)\Omega_y \\ \sin(\theta)\Omega_x + \cos(\theta)\Omega_y \\ 0 \end{bmatrix}.$$

Neglect cross-axis effects ($O(\Omega^2/\omega_0^2)$, like centrifugal effect).

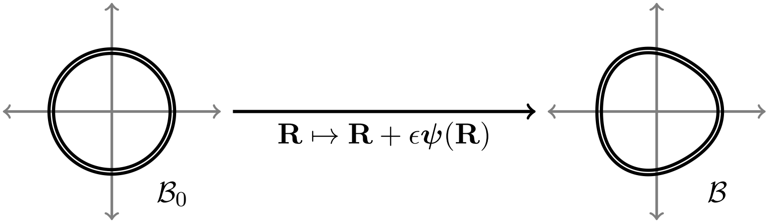
Analysis in Ideal Case

Only need to mesh a 2D cross-section!

- Compute an operating mode $\mathbf{u}_c$ for the non-rotating geometry.
- Compute $\beta = b(\mathbf{u}_c, e_z \times \mathbf{u}_s)$.
- Model: motion is approximately $q_1 \mathbf{u}_c + q_2 \mathbf{u}_s$, and

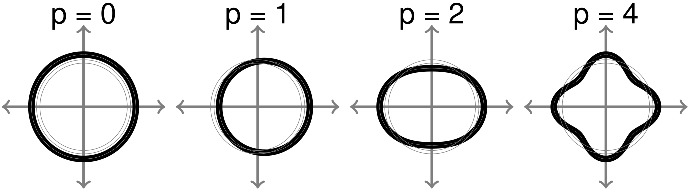
$$\ddot{\mathbf{q}} + 2\beta\Omega\mathbf{J}\dot{\mathbf{q}} + \omega_0^2 \mathbf{q} = 0,$$

Representing Imperfection



Write Fourier series for ψ, too!

Typical Fabrication Imperfections



Block matrix structure

Ex: $p = 2$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0 & & \epsilon & & \epsilon^2 & & \epsilon^3 & \\ & \mathbf{K}_1 & & \epsilon & & \epsilon^2 & & \\ \epsilon & & \mathbf{K}_2 & & \epsilon & & \epsilon^2 & \\ & \epsilon & & \mathbf{K}_3 & & \epsilon & & \\ \epsilon^2 & & \epsilon & & \mathbf{K}_4 & & \epsilon & \\ & \epsilon^2 & & \epsilon & & \mathbf{K}_5 & & \\ \epsilon^3 & & \epsilon^2 & & \epsilon & & \mathbf{K}_6 & \\ & & & & & & & \ddots \end{bmatrix}$$

Qualitative Information

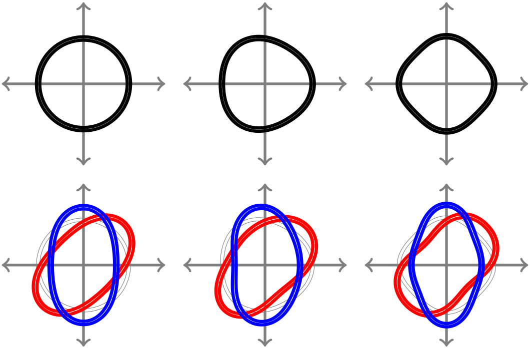
Operating wave number m, perturbation number p:

$p = 2m$	frequencies split by $O(\epsilon)$
$kp = 2m$	frequencies split at most $O(\epsilon^2)$
$p \not| 2m$	frequencies change at $O(\epsilon^2)$, *no* split
$p = 1$ $p = 2m \pm 1$	$O(\epsilon)$ cross-axis coupling.

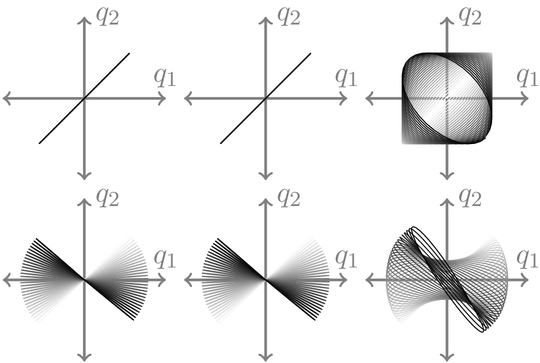
Note:

- $m = 2$ affected at first order by $p = 0$ and $p = 4$ (and $O(\epsilon^2)$) split from $p = 1$ and $p = 2$).
- $m = 3$ affected at first order by $p = 0$ and $p = 6$ (and $O(\epsilon^2)$) split from $p = 1$ and $p = 3$).

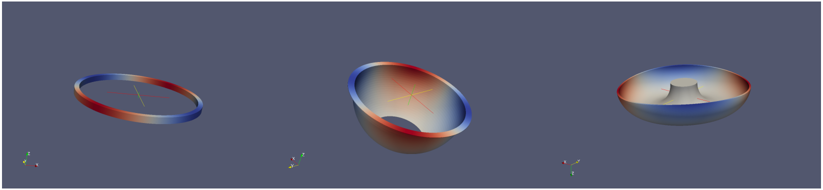
Analyzing Imperfect Rings



Analyzing Imperfect Rings



Read All About It!

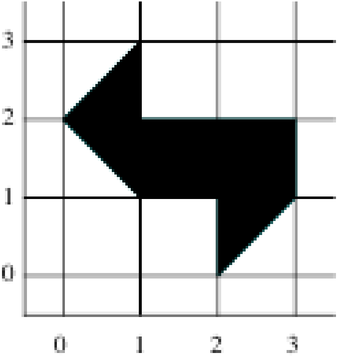


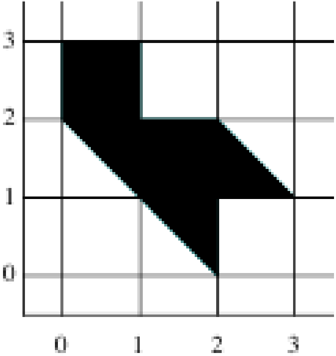
Text

Yilmaz and Bindel
"Effects of Imperfections on Solid-Wave Gyroscope Dynamics"
Proceedings of IEEE Sensors 2013, Nov 3–6.

Thanks to DARPA MRIG + Sunil Bhave and Laura Fegely.

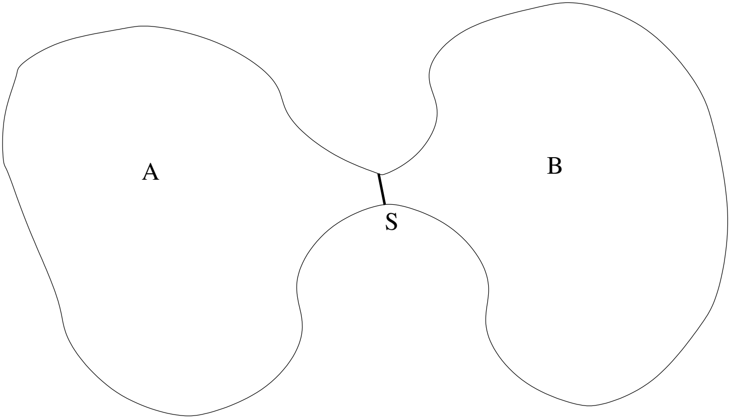
Can One Hear the Shape of a Drum?





$$-\nabla^2 u = \lambda u \text{ on } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

What Do You Hear?



Size of bottlenecks (Cheeger inequality)

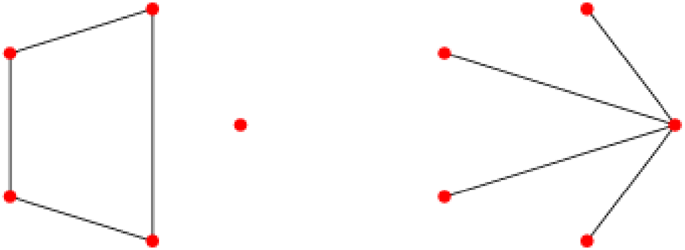
$$h \leq 2\sqrt{\lambda_2}$$

What Do You Hear?

Volume (Weyl law)

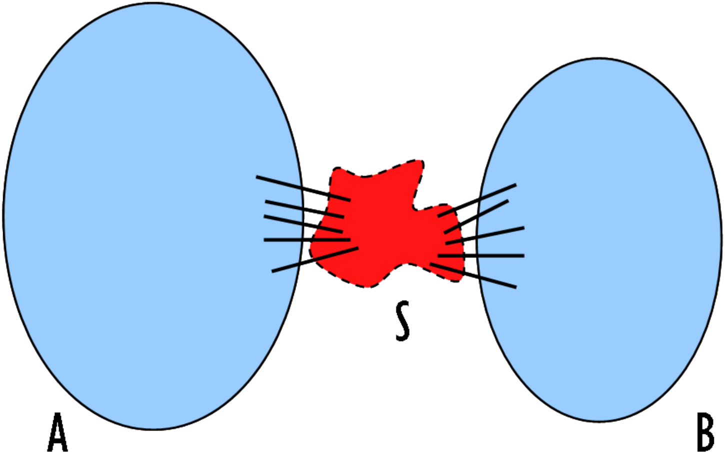
$$\lim_{x\to\infty} \frac{N(x)}{x^{d/2}} = (2\pi)^{-d}\omega_d \operatorname{vol}(\Omega), \quad N(x) = \{\text{\# eigenvalues } \le x\}$$

Can One Hear the Shape of a Graph?



From eigenvalues of adjacency, Laplacian, normalized Laplacian?

What Do You Hear?



Size of separators (Cheeger inequality)

What Do You Hear?

What information hides in the eigenvalue distribution?

1. Discretizations of Laplacian: something like Weyl's law
2. Sparse random graphs: Wigner semicircular distribution
3. "Real" networks: less well understood

But computing all eigenvalues seems *expensive*!

Exploring Spectral Densities

Kernel polynomial method (see Weisse, Reviews of Modern Physics)

- Think of spectral distribution as a generalized function

$$\int_{-1}^{1} \mu(x) f(x)\, dx = \frac{1}{N} \sum_{k=1}^{N} f(\lambda_k)$$

- Write $f(x) = \sum_{j=1}^{\infty} c_j T_j(x)$ and $\mu(x) = \sum_{j=1}^{\infty} d_j \phi_j(x)$, where $\int_{-1}^{1} \phi_j(x) T_k(x)\, dx = \delta_{jk}$
- Estimate $d_j = \frac{1}{N} \operatorname{tr}(T_j(A)) = \frac{1}{N} E[z^T T_j(A) z]$, z a random probe vector
- Truncate series for $\mu(x)$ and filter (avoid Gibbs)

Much cheaper than computing all eigenvalues!

Exploring Spectral Densities (with David Gleich)

- Consider spectrum of normalized Laplacian (random walk matrix)
- Approximate via KPM and compare to full eigencomputation

Things we know

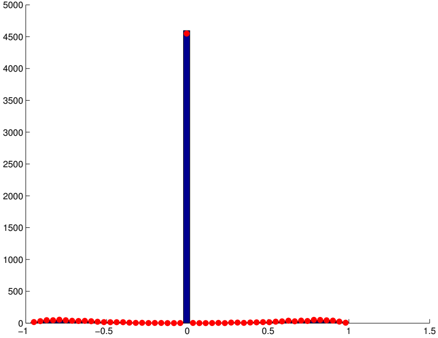
- Eigenvalues in $[-1, 1]$; nonsymmetric in general
- Stability: change d edges, have

$$\lambda_{j-d} \leq \hat{\lambda}_j \leq \lambda_{j+d}$$

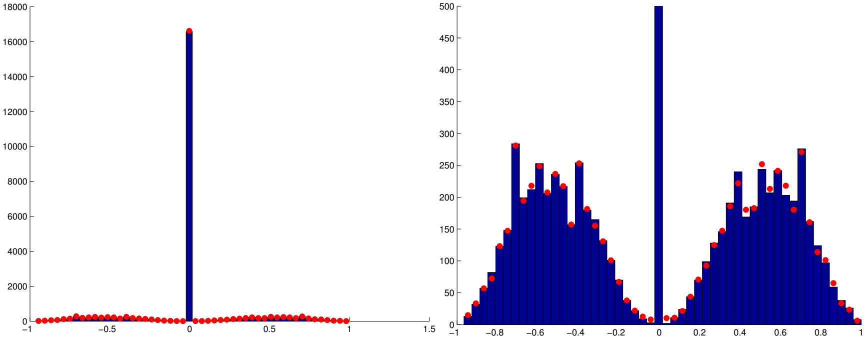
- kth moment = probability of return after k-step random walk
- Eigenvalue cluster near 1 $\sim$ well-separated clusters
- Eigenvalue cluster near 0 $\sim$ triangles connected by one node

What else can we "hear"?

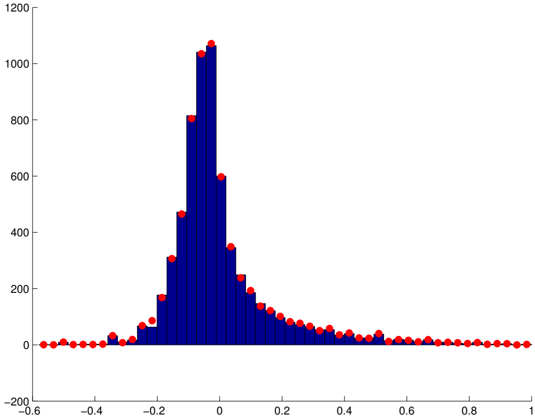
Erdos


5000
4500
4000
3500
3000
2500
2000
1500
1000
500
0
−1
−0.5
0
0.5
1
1.5
200
180
160
140
120
100
80
60
40
20
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

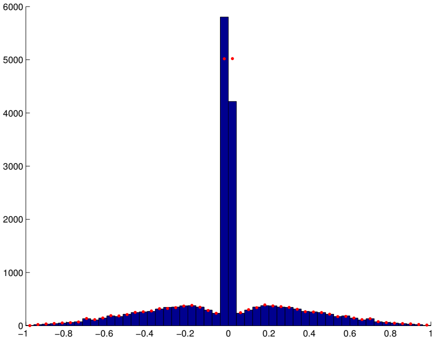
Internet topology

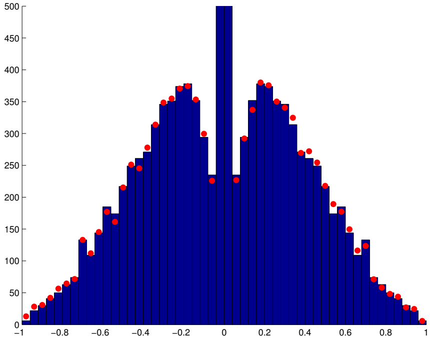


Marvel characters

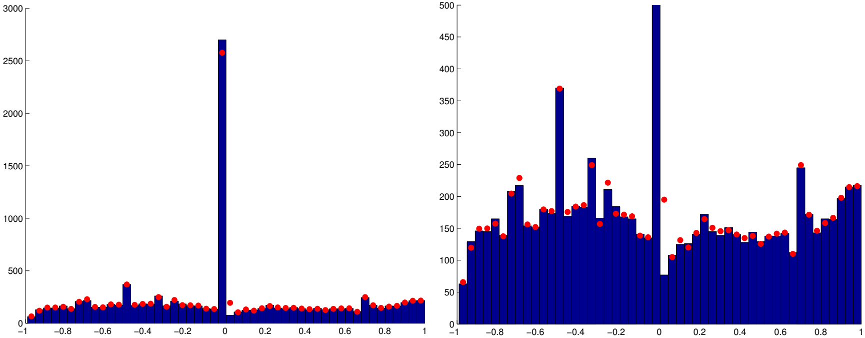


Marvel comics



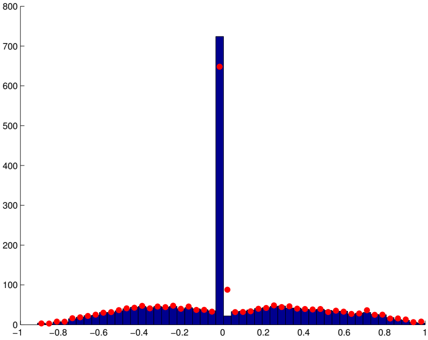


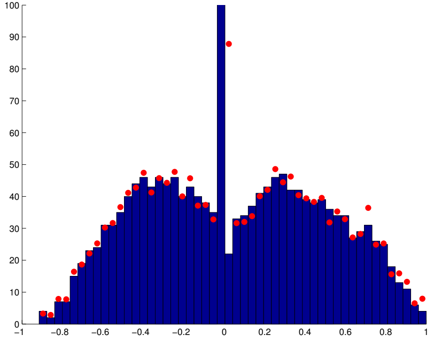
PGP



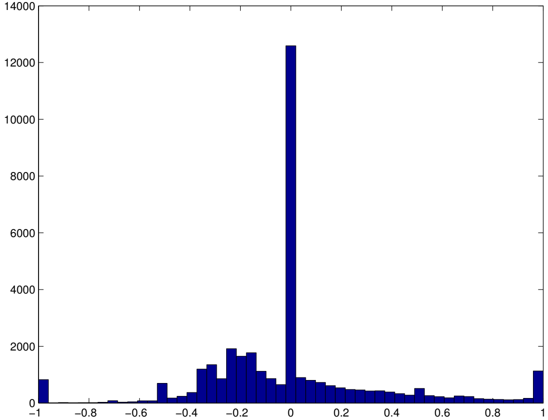
3000
2500
2000
1500
1000
500
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
500
450
400
350
300
250
200
150
100
50
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Yeast

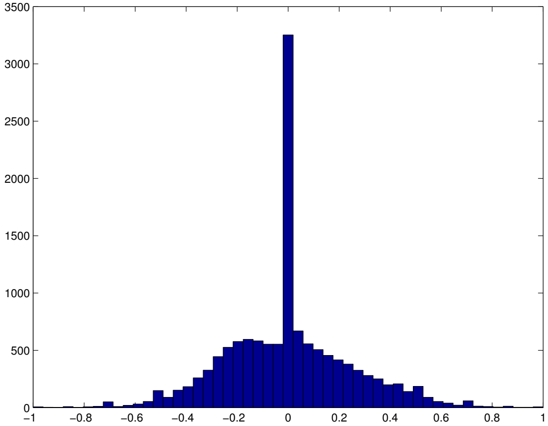

800
700
600
500
400
300
200
100
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


100
90
80
70
60
50
40
30
20
10
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

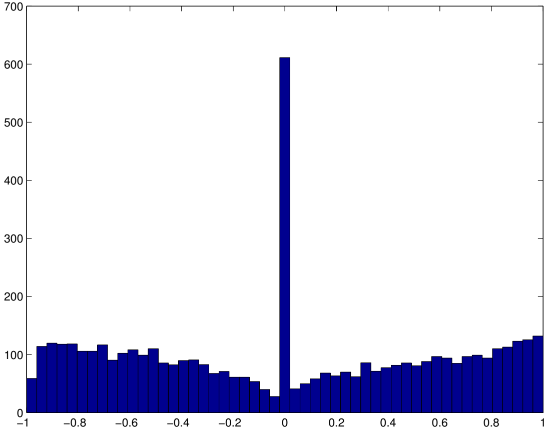
Enron emails (SNAP)



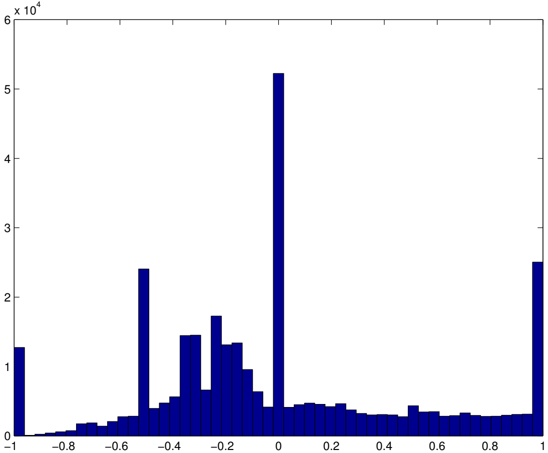
Reuters911 (Pajek)



US power grid (Pajek)

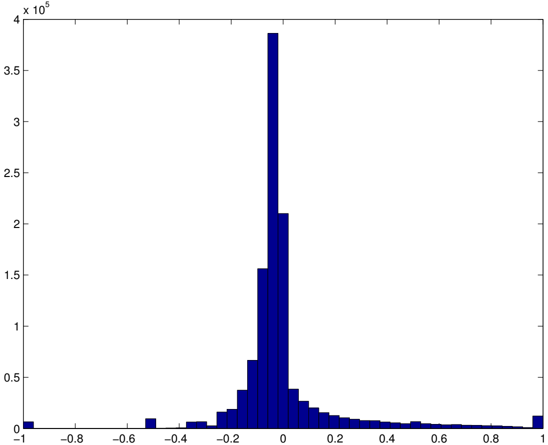


DBLP 2010 (LAW)



$N = 326186$, $nnz = 1615400$, 80 s (1000 moments, 10 probes)

Hollywood 2009 (LAW)



$N = 1139905$, $nnz = 113891327$, 2093 s (1000 moments, 10 probes)

What Do You Hear?