Music of the Microspheres
Eigenvalue problems from micro-gyro design

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G. H. Bryan (1864–1928)
Elected a Fellow of the Royal Society (1895)
Best known for *Stability in Aviation* (1911)
Also did important work in thermodynamics and hydrodynamics
Bryan was a friendly, kindly, very eccentric individual...

(Obituary Notices of the Fellows of the Royal Society)

... if he sometimes seemed a colossal buffoon, he himself did not help matters by proclaiming that he did his best work under the influence of alcohol.

(Williams, J.G., The University College of North Wales, 1884–1927)
Bryan’s Experiment

“On the beats in the vibrations of a revolving cylinder or bell” by G. H. Bryan, 1890
Bryan’s Experiment Today
The Beat Goes On
Free vibrations in a rotating frame (simplified):

\[ \ddot{\mathbf{q}} + 2\beta \Omega \mathbf{J} \dot{\mathbf{q}} + \omega_0^2 \mathbf{q} = 0, \quad \mathbf{J} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

Eigenvalue problem: \((-\omega^2 I + 2i\omega \beta \Omega \mathbf{J} + \omega_0^2) \mathbf{q} = 0\).

Solutions: \(\omega \approx \Omega_0 \pm \beta \Omega\). \(\implies\) beating \(\propto \Omega\)!
A Small Application

Northrup-Grummond HRG
(developed c. 1965–early 1990s)
A Smaller Application (Cornell)
A Smaller Application (UMich, GA Tech, Irvine)
A Smaller Application!
Micro-HRG / GOBLiT / OMG

Goal: Cheap, small (1mm) HRG

Collaborator roles:
- Basic design
- Fabrication
- Measurement

Our part:
- Detailed physics
- Fast software
- Sensitivity
- Design optimization
Rate Integrating Mode
Rate Integrating Mode
A General Picture

\[ \ddot{q} + 2\beta\Omega J\dot{q} + \omega_0^2 q = 0, \]

\[ J \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]
A General Picture

\[ \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \approx \begin{bmatrix} \cos(-\beta \Omega t) & -\sin(-\beta \Omega t) \\ \sin(-\beta \Omega t) & \cos(-\beta \Omega t) \end{bmatrix} \begin{bmatrix} q_1^0(t) \\ q_2^0(t) \end{bmatrix}. \]
An Uncritical FEA Approach

Why not do the obvious?

- Build 3D model with commercial FE
- Run modal analysis
The Perturbation Picture

Perturbations split degenerate modes:

- **Coriolis forces** (good)
- **Imperfect fab** (bad, but physical)
- **Discretization error** (non-physical)
Three Step Program

1. Perfect geometry, no rotation
2. Perfect geometry, rotation
3. Imperfect geometry
Step I: Perfect Geometry, No Rotation
Step I: Perfect Geometry, No Rotation

Free vibration problem in weak form

\[ \forall \mathbf{w}, \quad b(\mathbf{w}, \ddot{\mathbf{u}}) + a(\mathbf{w}, \mathbf{u}) = 0. \]

Symmetry: \( Q \) any rotation or reflection

\[ b(Q\mathbf{w}, Q\mathbf{u}) = b(\mathbf{w}, \mathbf{u}) \]
\[ a(Q\mathbf{w}, Q\mathbf{u}) = a(\mathbf{w}, \mathbf{u}) \]

Decompose by invariant subspaces of \( Q \implies \) Fourier analysis!
Fourier Expansion and Axisymmetric Shapes

Decompose into symmetric $u^c$ and antisymmetric $u^s$ in $y$:

$$u^c = \sum_{m=0}^{\infty} \Phi_m^c(\theta) u^c_m(r, z), \quad u^s = \sum_{m=0}^{\infty} \Phi_m^s(\theta) u^s_m(r, z)$$

where

$$\Phi_m^c(\theta) = \text{diag} (\cos(m\theta), \sin(m\theta), \cos(m\theta))$$

$$\Phi_m^s(\theta) = \text{diag} (-\sin(m\theta), \cos(m\theta), -\sin(m\theta)).$$

Modes involve only one azimuthal number $m$; degenerate for $m > 1$.

Preserve structure in FE: shape functions $N_j(r, z) \Phi_m^{c,s}(\theta)$
Finite element system: $M\ddot{u}^h + Ku^h = 0$

$$K = \begin{bmatrix} K_{0}^{cc} & K_1^{ss} & K_1^{cc} & K_2^{ss} & K_2^{cc} & \cdots & K_M^{ss} \\ 0 & K_1^{cc} & K_2^{cc} & \cdots & K_M^{cc} \end{bmatrix}$$

Mass has same structure.
Step II: Perfect Geometry, Rotation

Free vibration problem in weak form

$$\forall w, \quad b(w, a) + a(w, u) = 0.$$  

where

$$a = \ddot{u} + 2\Omega \times \dot{u} + \Omega \times (\Omega \times x) + \dot{\Omega} \times x$$

Discretize by finite elements as before:

$$M\ddot{u}^h + C\dot{u}^h + K\dot{u}^h = 0$$

where $C$ comes from Coriolis term ($2b(w, \Omega \times \dot{u})$).
Discretize $2b(w, \Omega \times \dot{u})$:

$$
C = \begin{bmatrix}
C_{00} & C_{01} \\
C_{10} & C_{11} & C_{12} \\
C_{21} & C_{22} & C_{23} \\
& & \ddots & \ddots & \ddots
\end{bmatrix}
$$

Off-diagonal blocks come from cross-axis sensitivity:

$$
\Omega = \Omega_z e_z + \Omega_{r\theta} = \begin{bmatrix}
0 \\
0 \\
\Omega_z
\end{bmatrix} + \begin{bmatrix}
\cos(\theta)\Omega_x - \sin(\theta)\Omega_y \\
\sin(\theta)\Omega_x + \cos(\theta)\Omega_y \\
0
\end{bmatrix}.
$$

Neglect cross-axis effects ($O(\Omega^2/\omega_0^2)$, like centrifugal effect).
Analysis in Ideal Case

Only need to mesh a 2D cross-section!

- Compute an operating mode $u_c$ for the non-rotating geometry.
- Compute associated modal mass and stiffness $m$ and $k$.
- Compute $g = b(u_c, e_z \times u_s)$.
- Model: motion is approximately $q_1 u_c + q_2 u_s$, and

$$m\ddot{q} + 2g\Omega J\dot{q} + kq = 0,$$
Step III: Imperfect Geometry
What Imperfections?

Let me count the ways...

- Over/under etch
- Mask misalignment
- Thickness variations
- Anisotropy of etching single-crystal Si

These are *not* arbitrary!
Representing the Perturbation

Map axisymmetric $\mathcal{B}_0 \rightarrow$ real $\mathcal{B}$:

$$ R \in \mathcal{B}_0 \mapsto r = R + \epsilon \psi(R) \in \mathcal{B}. $$

Write weak form in $\mathcal{B}_0$ geometry:

$$ b(w, a) = \int_{\mathcal{B}_0} \rho w \cdot a J d\mathcal{B}_0, $$

$$ a(w, u) = \int_{\mathcal{B}_0} \epsilon(w) : C : \epsilon(u) J d\mathcal{B}_0, $$

where $J = \det(I + \epsilon F)$ with $F = \partial \psi / \partial R$. 
Decomposing $\psi$

Do Fourier decomposition of $\psi$, too! Consider case where

\[ m = \text{only azimuthal number of } w \]
\[ n = \text{only azimuthal number of } u \]
\[ p = \text{only azimuthal number of } \psi \]

Then we have selection rules

\[ a(w, u) = \begin{cases} 
O(\epsilon^k), & |m - n| = kp \\
0, & \text{otherwise}
\end{cases} \]

Similar picture for $b$. 
Decomposing $\psi$

- Over/under etch ($p = 0$)
- Mask misalignment ($p = 1$)
- Thickness variations ($p = 1$)
- Anisotropy of etching single-crystal Si ($p = 4$)
Block matrix structure

Ex: $p = 2$

$$K = \begin{bmatrix} K_0 & \epsilon & \epsilon^2 & \epsilon^3 \\ \epsilon & K_1 & \epsilon & \epsilon^2 \\ \epsilon^2 & \epsilon & K_2 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon & K_3 \\ \epsilon^2 & \epsilon & K_4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon & K_5 \\ & & \ddots & \ddots \end{bmatrix}$$
Impact of Selection Rules

- Fast FEA: Can neglect some wave numbers / blocks
- Also *qualitative* information
Qualitative Information

Operating wave number $m$, perturbation number $p$:

<table>
<thead>
<tr>
<th>$p = 2m$</th>
<th>frequencies split by $O(\epsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$kp = 2m$</td>
<td>frequencies split at most $O(\epsilon^2)$</td>
</tr>
<tr>
<td>$p \nmid 2m$</td>
<td>frequencies change at $O(\epsilon^2)$, no split</td>
</tr>
</tbody>
</table>

| $p = 1$ | $p = 2m \pm 1$ | $O(\epsilon)$ cross-axis coupling. |

Note:

- $m = 2$ affected at first order by $p = 0$ and $p = 4$ (and $O(\epsilon^2)$ split from $p = 1$ and $p = 2$).
- $m = 3$ affected at first order by $p = 0$ and $p = 6$ (and $O(\epsilon^2)$ split from $p = 1$ and $p = 3$).
Mode Split for Rings: \( \psi(r, \theta) = (\cos(2m\theta), 0) \).
Mode Split for Rings: \( \psi(r, \theta) = (\cos(m\theta), 0) \).
Analyzing Imperfect Rings
Analyzing Imperfect Rings
Beyond Rings: AxFEM

- Mapped finite / spectral element formulation
- Low-order polynomials through thickness
- High-order polynomials along length
- Trig polynomials in $\theta$
- Agrees with results reported in literature
- Computes sensitivity to geometry, material parameters, etc.
Further Steps

Lots of possible directions:

- Symmetry breaking through damping?
- Integration with fabrication simulation?
- Joint optimization of geometry and fabrication?
Thank You

Yilmaz and Bindel
“Effects of Imperfections on Solid-Wave Gyroscope Dynamics”

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