Refining Approximate Invariant Subspaces

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Overview

- Problem and motivation
- Representing invariant subspaces
- The Riccati equation
- Eigensolvers as equation solvers
- Summary and concluding thoughts
Problem statement

• Input:
  • Matrix $A$
  • Approximate invariant subspace $V_0$

• Output:
  • Improved approximation $V$
Judging approximate subspaces

• Eigenpair residual equation is

\[ R(v, \lambda) = Av - v\lambda = 0 \]

• Invariant subspace residual equation is

\[ R(V, L) = AV - VL = 0 \]
Motivating examples

Numerically solve \( x'(t) = f(x) \) at \( t_1, t_2, \ldots \).

Interested in modes of linearized system

\[
\frac{dx}{d\tau}(t_i + \tau) \approx A_i(x(t_i + \tau) - x(t_i))
\]

\[
A_i := \frac{\partial f}{\partial x}(x(t_i))
\]

about a sequence of operating points. Useful for

- Local reduced models
- Superimposed vibrations (sound)
- Stabilization of integrators
Motivating examples

Following path of equilibrium solutions \( \hat{x}(p) \) of parameterized ODE

\[
x'(t) = f(x; p)
\]

Interested in stability of the equilibrium at a sequence of \( p_i \). Follow eigenvalues of

\[
A_i = \frac{\partial f}{\partial x}(\hat{x}(p_i))
\]

to see if any cross imaginary axis.
Motivating examples

- Approximating a PDE with multigrid
- Interested in a few modes
- $A_1, A_2, \ldots$ are increasingly fine discretizations.
Summary: Residual equations

Wrote two residual equations:

\[ R(V, L) = AV - VL \text{ where } V_0^*V = I \]
\[ R(U) = A_{12} + A_{22}U - UA_{11} - UA_{12}U \]

Applied standard nonlinear solver techniques. Residual functions also exist for generalized and polynomial problems.
Classifying eigensolvers

- Which residual? Unknown is \((V, L)\) or \(U\)?
- What constraint isolates solutions?
- Symmetric or nonsymmetric?
- Standard, generalized, or nonlinear?
- Dense or sparse (projection based)?
- Choice of projection space(s)?
  - How to expand and contract?
  - Preconditioning?
- What nonlinear solver iteration?