Example 1

Let X and Y be independant random variables, each uniformly distributed on the interval $(0, A)$. Let $V = max(X, Y)$. Lets find the p.d.f. and c.d.f. of V. We have already seen that the p.d.f. of a uniformly distributed r.v.

$$f(x) = \begin{cases} \frac{1}{interval\ length} & x \in interval \\ 0 & otherwise \end{cases}$$

and in our case

$$f_X(x) = \begin{cases} \frac{1}{A} & x \in (0, A) \\ 0 & otherwise \end{cases}$$

To get the c.d.f. of X we must integrate its p.d.f. to get

$$F_X(x) = \int_0^x \frac{1}{A} du = \frac{1}{A} u]_0^x = \frac{1}{A}(x - 0) = \frac{x}{A}$$

Now if $V = max(X, Y) \leq v$ then both $X \leq v$ and $Y \leq v$, so

$$F_V(v) = P(V \leq v) = P(X \leq v\ and\ Y \leq v)$$

Since X and Y are independent we can write

$$F_V(v) = P(V \leq v) = P(X \leq v)\ P(Y \leq v) = F_X(v)\ F_Y(v) = \frac{v}{A} * \frac{v}{A} = \frac{v^2}{A^2}$$

To get the p.d.f. of V we must difrentiate its c.d.f. and get

$$f_V(v) = \frac{d}{dv} F_V(v) = \frac{d}{dv} \frac{v^2}{A^2} = \frac{2v}{A^2}$$

Example 2

Let X and Y be independant random variables, each exponentially distributed with parameter $\lambda = 1$ and let $W = X - Y$.
Lets find the p.d.f. and c.d.f. of W.
For any value w of $W = X - Y$ there are many options of what values X and Y can get. We can be more percise and determine that for any value w, if $X = x$ then Y must equal $(x - w)$ so that $W = X - Y = x - (x - w) = w$.
Lets look at the p.d.f. of W when $w \geq 0$. Notice that X and Y are independant random variables.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - w) dx$$

Since both x and $y = (x - w)$ must be greater than 0 for the p.d.f.'s to be non-zero, we can change the integration limits to

$$f_W(w) = \int_{w}^{\infty} f_X(x) f_Y(x - w) dx = \int_{w}^{\infty} e^{-x} * e^{-(x-w)} dx = \int_{w}^{\infty} e^{w} * e^{-2x} dx$$

$$= e^{w} \int_{w}^{\infty} e^{-2x} dx = e^{w} * \frac{-1}{2} * e^{-2x}]_{w}^{\infty} = e^{w} * [0 - (-\frac{-1}{2} * e^{-2w}] = e^{w} * \frac{-1}{2} e^{-2w} = \frac{1}{2} e^{-w}$$

Since $W = X - Y$ and Y isn't necessarily lessesr than X, W might get negative values. So lets examine the p.d.f. of W where $w \leq 0$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - w) dx$$

Still, both x and $y = (x - w)$ must be greater than 0 for the p.d.f.'s to be non-zero, but $x - w \geq x$ since $w \leq 0$ and our inegration limits change to $x \geq 0$:

$$f_W(w) = \int_{0}^{\infty} f_X(x) f_Y(x - w) dx = \int_{0}^{\infty} e^{-x} * e^{-(x-w)} dx = \int_{0}^{\infty} e^{w} * e^{-2x} dx$$

$$= e^{w} \int_{0}^{\infty} e^{-2x} dx = e^{w} * \frac{-1}{2} * e^{-2x}]_{0}^{\infty} = e^{w} * \frac{-1}{2} * [0 - 1] = \frac{1}{2} e^{w}$$

To get the c.d.f. of W we must integrate its p.d.f. to get $F_W(w) = \int_{-\infty}^{w} f_W(v) dv$.
Again we must look at 2 sections depending on the sign of w.
For $w \leq 0$ we get:

$$F_W(w) = \int_{-\infty}^{w} f_W(v) dv = \int_{-\infty}^{w} \frac{1}{2} e^{v} dv = \frac{1}{2} \int_{-\infty}^{w} e^{v} dv = \frac{1}{2} e^{v}]_{-\infty}^{w} = \frac{1}{2}(e^{w} - 0) = \frac{1}{2} e^{w}$$

While for $w \geq 0$ we get:

$$F_W(w) = \int_{-\infty}^{w} f_W(v) dv = \int_{-\infty}^{0} \frac{1}{2} e^{v} dv + \int_{0}^{w} \frac{1}{2} e^{-v} dv = \frac{1}{2} e^{v}]_{-\infty}^{0} + \frac{1}{2} e^{-v}]_{0}^{w}$$

$$= \frac{1}{2}[1 - 0] + \frac{1}{2}(1 - e^{-w}) = 1 - \frac{1}{2} e^{-w}$$