Wireless Localization with Vertex Guards

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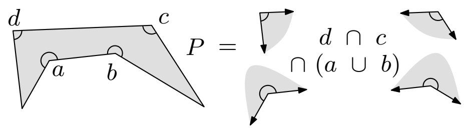
Abstract

We consider the *wireless localization problem.* Given a simple polygon P, place and orient guards each of which broadcasts a unique key within a fixed angular range. At any point in the plane one must be able to tell whether or not one is located inside P only by looking at the set of keys received. In other words, the interior of the polygon must be described by a monotone Boolean formula composed from the guards. We improve the upper bound for the vertex guard problem where guards may be placed on vertices of P only and show that the maximum number of vertex guards needed to describe any simple polygon on n vertices is at most $\frac{8}{9}n$.

1 Wireless Localization

Art gallery problems are a classic topic in discrete and computational geometry. A new direction has been introduced by Eppstein, Goodrich, and Sitchinava [4]. They propose to modify the concept of visibility by not considering the edges of the polygon/gallery as blocking. This changes the problem drastically. The motivation for this model stems from communication in wireless networks where the signals are not blocked by walls, either. For illustration, suppose you run a café (modeled as a simple polygon P) and you want to provide wireless internet access. But you do not want the whole neighborhood to use your infrastructure. Instead, Internet access should be limited to those people who are located within the café. To achieve this, you can install a certain number of devices, let us call them guards, each of which broadcasts a unique (secret) key in an arbitrary but fixed angular range. The goal is to place guards and adjust their angles in such a way that everybody who is inside the café can prove this fact just by naming the keys received and nobody who is outside the café can provide such a proof. Formally this means that P must be described by a monotone Boolean formula over the keys, that is, a formula using the operators AND and OR only, negation is not allowed. It is convenient to model a guard as a subset of the plane, namely the area where the broadcast from this guard can be received. This



area can be described as an intersection or union of at most two halfplanes. Using this notation, the polygon P is to be described by a combination of the operations union and intersection over the guards. For example, the polygon P above can be described by $(a \cup b) \cap c \cap d$.

Natural guards. Natural locations for guards are the vertices and edges of P. A guard which is placed at a vertex of P is called a *vertex guard.* A vertex guard is *natural* if it covers exactly the interior angle of its vertex. Natural vertex guards alone do not always suffice [4]. A guard placed anywhere on the line given by an edge of P and broadcasting within an angle of π to the inner side of the edge is called a *natural edge guard.* Dobkin, Guibas, Hershberger, and Snoeyink [3] showed that n natural edge guards are sufficient for any simple polygon with n edges. Using both natural vertex guards and natural edge guards, $n-2$ guards are sufficient and can be necessary [1].

Vertex guards. Using a different approach, Eppstein et al. [4] proved that any simple polygon with n edges can be guarded using at most $n-2$ (general, that is, not necessarily natural) vertex guards. In this work, we improve the upper bound to $\lfloor \frac{8n-6}{9} \rfloor$ for $n \geq 4$. This bound is still not known to be tight. Damian, Flatland, O'Rourke, and Ramaswami [2] describe a family of simple polygons with n edges which require at least $\lfloor 2n/3 \rfloor - 1$ vertex guards.

General guards. In the most general setting, we do not have any restriction on the placement and the angles of guards. At the moment, the best known upper bound is $\lfloor \frac{4n-2}{5} \rfloor$, which is not known to be tight, the best lower bound being $\lceil \frac{3n-4}{5} \rceil$ [1].

The different problems and results are summarized in the following table. The mark $*$ indicates the result of this paper.

guards	lower bound	upper bound
natural	$n-2$ [1]	$n-2$ [3]
vertex	$\lfloor 2n/3 \rfloor - 1$ [2]	$\lfloor (8n-6)/9 \rfloor$ [$*$]
general	$\lceil (3n-4)/5 \rceil$ [1]	$\lfloor (4n-2)/5 \rfloor$ [1]

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2 Upper Bound for Vertex Guards

We use the notion of a *polygonal halfplane* H which is a topological halfplane bounded by a *simple bi-infinite polygonal chain* $C = (e_1, \ldots, e_n)$, for a positive integer n. For $n = 1$, the only edge e_1 is a line and the polygonal halfplane is a halfplane. For $n = 2$, e_1 and e_2 are rays which share a common source but are not collinear. For $n \geq 3$, e_1 and e_n are rays, e_i is a line segment, for $1 < i < n$, and e_i and e_j, for $1 \leq i < j \leq n$, do not intersect unless $j = i+1$ in which case they share an endpoint. For brevity we use the term *chain* in place of simple bi-infinite polygonal chain. Let v_i, for $1 \leq i < n$, denote the vertex of C incident to e_i and e_{i+1}, $V(C) := \{v_1, \ldots, v_{n-1}\}$. For $2 \leq i \leq n-1$, let e_i^+ be the ray obtained from e_i by extending the segment linearly beyond v_i. Similarly e_i^- refers to the ray obtained from e_i by extending the segment linearly beyond v_{i-1}. For a polygonal halfplane H define $\gamma(H)$ to be the minimum integer k such that there exists a guarding $\mathcal{G}(H)$ for H using k vertex guards. Similarly, for a natural number n, denote by $\gamma(n)$ the maximum number $\gamma(H)$ such that H is bounded by a chain with n edges. Obviously, $\gamma(1) = \gamma(2) = 1$. Observe that any guarding for H can be transformed into a guarding for the complement $\overline{H}$ using the same number of guards: Use de Morgan's rules and invert all guards (keep their location but flip the angle to the complement with respect to 2π). Therefore, we can define $\gamma(C) = \gamma(H) = \gamma(\overline{H})$.

Theorem 1 *For any $n \geq 2$, $\gamma(n) \leq \lfloor \frac{8n-3}{9} \rfloor$.*

Proof. The base cases $\gamma(1) = \gamma(2) = 1$, $\gamma(3) = 2$, $\gamma(4) = 3$, $\gamma(5) = 4$ and $\gamma(6) = 5$ follow from the observation that a chain can always be guarded with $n-1$ natural guards [1]. Now let H be a polygonal halfplane bounded by an oriented polygonal chain C with $n \geq 7$ edges such that the interior of H lies to the right of C. Let $S := V(\mathrm{conv}(V(C)))$ be the vertices of the convex hull of $V(C)$, that is, the vertices of C that are extremal. The basic idea is to *split* C at a vertex $v_i \in S$ into two chains $C_1 = (e_1, \ldots, e_{i-1}, e_i^+)$ and $C_2 = (e_{i+1}^-, e_{i+2}, \ldots, e_n)$. If the "new" rays e_i^+ and e_{i+1}^- do not intersect the "old" rays e_1 and e_n, we can express H as the intersection or union of the two polygonal halfplanes H_1 and H_2 bounded by C_1 and C_2 depending on whether v_i is convex or reflex. Assume that the angle between e_1 and e_n is convex (else, consider $\overline{H}$ instead of H) and think of C as going from the left to the right (thus H being below C). In other words, e_1 and e_n are assumed to go from left to right, e_1 having positive slope and e_n having smaller slope than e_1. Now look at the convex hull $\mathrm{conv}(H)$ of H. There must be at least one vertex v_i in S which lies on the boundary $\partial\mathrm{conv}(H)$. Such a vertex is for sure a good splitting vertex in the above sense. If $2 \leq i \leq n-2$, we split C at v_i as explained into two chains $C_1 := (e_1, \ldots, e_{i-1}, e_i^+)$ and $C_2 := (e_{i+1}^-, e_{i+2}, \ldots, e_n)$ and get a guarding $\mathcal{G}(C_1) \cap \mathcal{G}(C_2)$, where $\mathcal{G}(C_i)$ denotes the guarding of C_i we get by induction, see Figure 1. Therefore $\gamma(C) \leq \gamma(i) + \gamma(n-i) \leq \lfloor \frac{8i-3}{9} \rfloor + \lfloor \frac{8(n-i)-3}{9} \rfloor \leq \lfloor \frac{8n-6}{9} \rfloor \leq \lfloor \frac{8n-3}{9} \rfloor$.

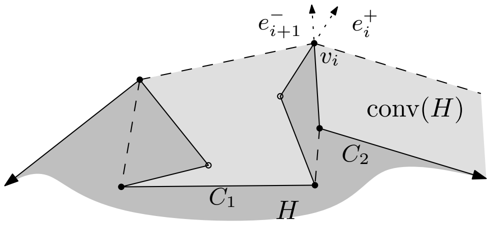

Figure 1: Splitting at a convex hull vertex.

If there is no vertex on $\partial\mathrm{conv}(H)$ with index $2 \leq i \leq n-2$, we first consider the case that $S \cap \partial\mathrm{conv}(H) = \{v_1, v_{n-1}\}$. If there is a vertex $v_i \in S$ with $3 \leq i \leq n-3$, split C at v_1, v_i and v_{n-1}: Put a natural edge guard g_1 onto e_1, a natural edge guard g_2 onto e_n and define $C_1 := (e_2^-, \ldots, e_i^+)$, $C_2 := (e_{i+1}^-, \ldots, e_{n-1}^+)$. (We can place the natural edge guards on the incident vertices, hence the natural edge guards can be realized as (non-natural) vertex guards.) Then, a guarding for C can be obtained as $g_1 \cap g_2 \cap (\mathcal{G}(C_1) \cup \mathcal{G}(C_2))$, see Figure 2. This implies $\gamma(C) \leq 2 + \gamma(i-1) + \gamma(n-i-1) \leq 2 + \lfloor \frac{8i-11}{9} \rfloor + \lfloor \frac{8(n-i)-11}{9} \rfloor \leq \lfloor \frac{8n-6}{9} \rfloor$.

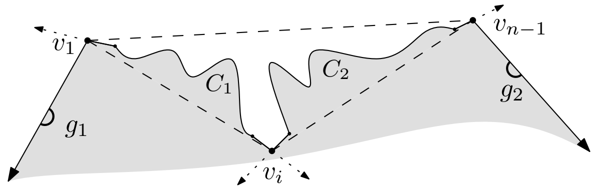

Figure 2: v_1 and v_{n-1} are the only vertices on $h(C)$.

If $S = \{v_1, v_2, v_{n-2}, v_{n-1}\}$, consider $S' := V(\mathrm{conv}\{v_2, \ldots, v_{n-2}\})$. Beside v_2 and v_{n-2}, there must be a third vertex $v_j \in S'$, without loss of generality $4 \leq j \leq n-3$ (if $j = 3$, reflect C). If e_3^- does not intersect e_n, put a natural edge guard g_1 onto e_1, a vertex guard g_2 onto v_2 with the right ray covering e_2 and its other ray parallel to e_n and a natural vertex guard g_3 onto v_{n-1} and define $C_1 := (e_3^-, \ldots, e_j^+)$ and $C_2 := (e_{j+1}^-, e_{j+2}, \ldots, e_{n-2}, r)$ where r is the ray starting at v_{n-2} in the direction of e_n. See Figure 3. Then we get a guarding as $g_1 \cap (g_2 \cup (\mathcal{G}(C_1) \cap \mathcal{G}(C_2)) \cup g_3)$ and conclude $\gamma(C) \leq 3 + \gamma(j-2) + \gamma(n-j-1) \leq 3 + \lfloor \frac{8(j-2)-3}{9} \rfloor + \lfloor \frac{8(n-j-1)-3}{9} \rfloor \leq \lfloor \frac{8n-3}{9} \rfloor$. If e_3^- intersects e_n, put a natural vertex guard g_1 onto v_1, and an edge guard g_2 onto e_n and define $C' := (e_3^-, \ldots, e_{n-1}^+)$. See Figure 3. We obtain a guarding $(g_1 \cup \mathcal{G}(C')) \cap g_2$.

If S consists of 3 vertices only and there is no $v_i \in S$ with $3 \leq i \leq n-3$, assume without loss of generality that $v_2 \in S$ (if v_{n-2} is the only vertex in S beside

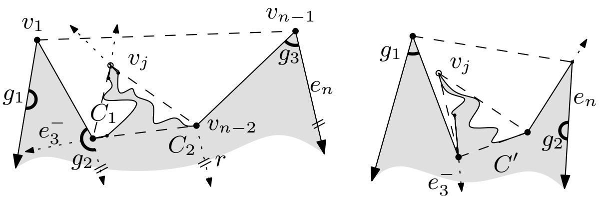

Figure 3: $S = \{v_1, v_2, v_{n-2}, v_{n-1}\}$

v_1 and v_{n-1}, reflect C). In this case, define $S' := V(\text{conv}(\{v_2, \ldots, v_{n-1}\}))$. For sure, $v_2, v_{n-1} \in S'$ but there must be a third vertex $v_j \in S'$, see Figure 4.

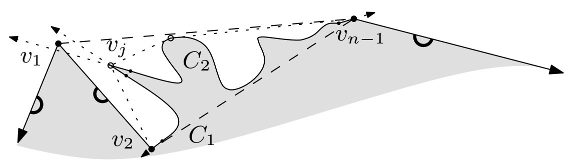

Figure 4: v_1, v_2, and v_{n-1} are the only vertices in S.

If $4 \leq j \leq n-3$, put a edge guards onto e_1, e_2, and e_n, and split the remaining chain at v_j. If v_3 is the only new vertex in S', put a natural vertex guard g_1 onto v_1, and a non-natural vertex guard g_2 onto v_2 covering e_3 with its left ray and with the right ray parallel to e_1, and an edge guard g_3 onto e_n: A guarding can be obtained as $g_3 \cap (g_1 \cup (g_2 \cap \mathcal{G}(C')))$ where $C' = (e_4^-, \ldots, e_{n-1}^+)$. See Figure 5.

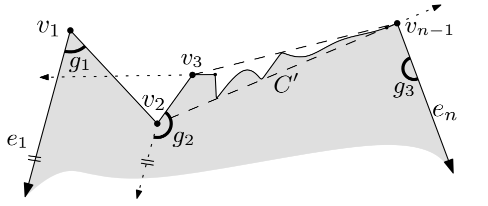

Figure 5: $S' = \{v_{n-1}, v_3, v_2\}$.

If v_{n-2} is the only new vertex, define $C' = (e_3^-, \ldots, e_{n-2}^+)$, put a natural vertex guard g_3 onto v_{n-1}. If e_3^- does not intersects e_n, put an edge guard g_1 onto e_1 and a vertex guard g_2 onto v_2 covering e_2 with its right ray and with its left ray parallel to e_n. We get a guarding $g_1 \cap (g_2 \cup (\mathcal{G}(C') \cap g_3))$. If e_3^- intersects e_n, put a natural vertex guard g_1 onto v_1 and an edge guard g_2 onto e_n and observe $H = g_2 \cap (g_1 \cup (\mathcal{G}(C') \cap g_3))$. We conclude $\gamma(C) \leq 3 + \gamma(n-4) \leq 3 + \lfloor \frac{8n-32-3}{9} \rfloor \leq \lfloor \frac{8n-8}{9} \rfloor$, see Figure 6.

Finally, assume there is only one vertex on $\partial\text{conv}(H)$, which is either v_1 or v_{n-1}. We assume without loss of generality that it is v_1. Beside v_1, which for sure belongs to S, there must be at least two more vertices in S. Let v_i be the vertex of S which is extremal to the right of e_n. If $3 \leq i \leq n-2$, split C into three parts cutting it at v_1 and v_i. Then, we get a guarding for C as $g \cap (\mathcal{G}(C_1) \cup \mathcal{G}(C_2))$, where

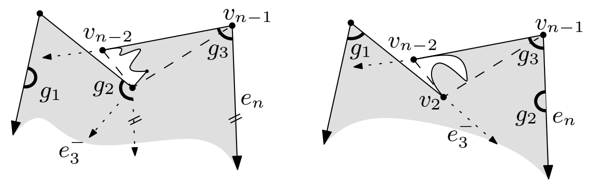

Figure 6: $S' = \{v_{n-1}, v_{n-2}, v_2\}$.

g is a natural edge guard on e_1, $C_1 = (e_2^-, \ldots, e_i^+)$, and $C_2 = (e_{i+1}-, \ldots, e_n)$, see Figure 7.

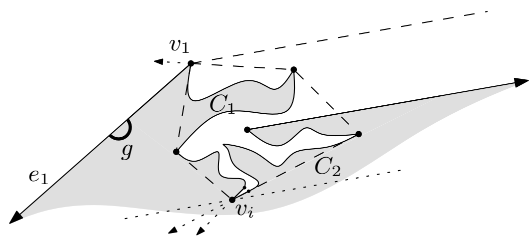

Figure 7: Split at the extremal vertex below e_n.

If $i = 2$, define $S' := V(\text{conv}(\{v_2, \ldots, v_{n-1}\}))$. Let v_j be the vertex of S' which is extremal in the opposite direction, that is, to the left of e_n. If $4 \leq j \leq n-2$, we put natural edge guards g_1 and g_2 onto e_1 and e_2 and split the rest at v_j into two chains C_1 and C_2, see Figure 8. Then $H = g_1 \cap (g_2 \cup (\mathcal{G}(C_1) \cap \mathcal{G}(C_2)))$.

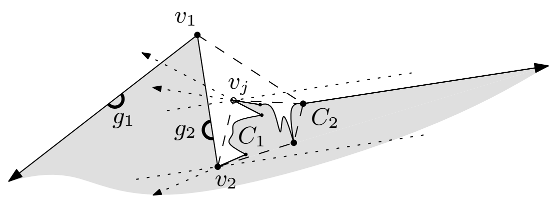

Figure 8: The extremal vertex in S below e_n is v_2.

If $j = 3$, we put a natural vertex guard onto v_1, a guard onto v_2 with its left ray covering e_3 and the right parallel to e_1. Then, we get $\gamma(C) \leq 2 + \gamma(n-3) \leq 2 + \lfloor \frac{8n-27}{9} \rfloor \leq \lfloor \frac{8n-9}{9} \rfloor$. If $j = n-1$, that is, if there is no vertex above e_n except v_1, take any vertex $v_s \in S'$, $3 \leq s \leq n-2$. If $4 \leq s \leq n-3$, put an edge guard g_1 onto e_1 and an edge guard g_2 onto e_2 and an edge guard g_3 onto e_n and define $C_1 = (e_3^-, \ldots, e_s^+)$, $C_2 := (e_{s+1}^-, \ldots, e_{n-1})$, then we get a guarding $g_1 \cap (g_2 \cup (g_3 \cap (\mathcal{G}(C_1) \cup \mathcal{G}(C_2))))$ (or $g_1 \cap (g_2 \cup (g_3 \cap \mathcal{G}(C_1) \cap \mathcal{G}(C_2)))$ if v_s is convex), see Figure 9. If $s = 3$, put 3 guards explicitly depending on whether v_3 is reflex or convex and cover the remaining chain $C' = (e_4^-, \ldots, e_{n-1}^+)$ recursively, see Figure 10. If $s = n-2$, we are in a situation similar to one of those shown in Figure 3 or 6 and proceed accordingly.

If $i = n-1$, that is, if there is no vertex of S below e_n, distinguish two cases: Either there is a convex vertex in S between v_{n-1} and v_1, or there is no such vertex. If there is a convex vertex $v_j \in S$, with

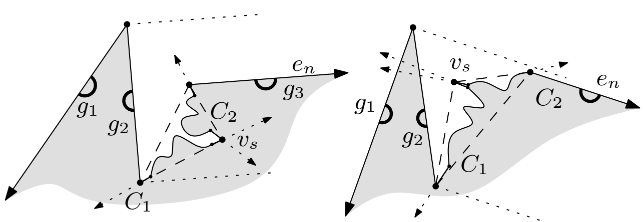

Figure 9: There is no vertex in S' above e_n.

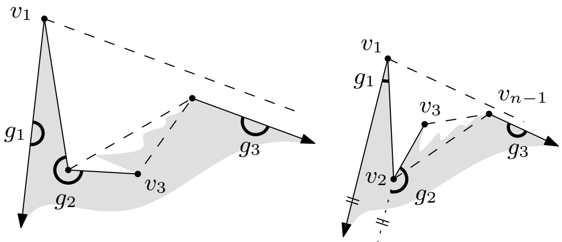

Figure 10: $S' = \{v_2, v_3, v_{n-1}\}$.

$2 \leq j \leq n-3$, put an edge guard onto e_n and split at v_j, see Figure 11. If $j = 2$ or $j = n-1$ proceed as above but now removing v_1 or v_{n-1}, respectively, defining $S' := V(\text{conv}(\{v_2, \ldots, v_{n-1}\}))$ ($S' := V(\text{conv}(\{v_1, \ldots, v_{n-3}\}))$, respectively), see Figure 12.

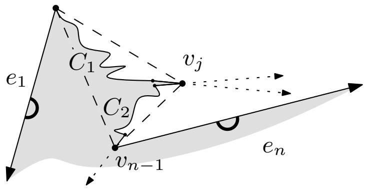

Figure 11: There is a convex vertex $v_j \in S$.

If $S = \{v_1, v_2, v_{n-1}\}$ and v_2 is reflex, let $S' = V(\text{conv}\{v_2, ..., v_{n-1}\})$ and look for a splitting vertex in S'. If there is $v_j \in S'$, $4 \leq j \leq n-3$, put edge guards onto e_1, e_2 and e_n and split the rest at j, see Figure 13. If $S' = \{v_2, v_{n-1}, v_3\}$ or $S' = \{v_2, v_{n-1}, v_3\}$, proceed as shown in Figure 14. $\square$

Lemma 2 (Lemma 4 in [1]) *A simple polygon P on $n \geq 4$ vertices is the intersection of two polygonal halfplanes both having at least two edges.*

Corollary 3 *A simple polygon P on $n \geq 4$ edges can be guarded with at most $\lfloor (8n-6)/9 \rfloor$ vertex guards.*

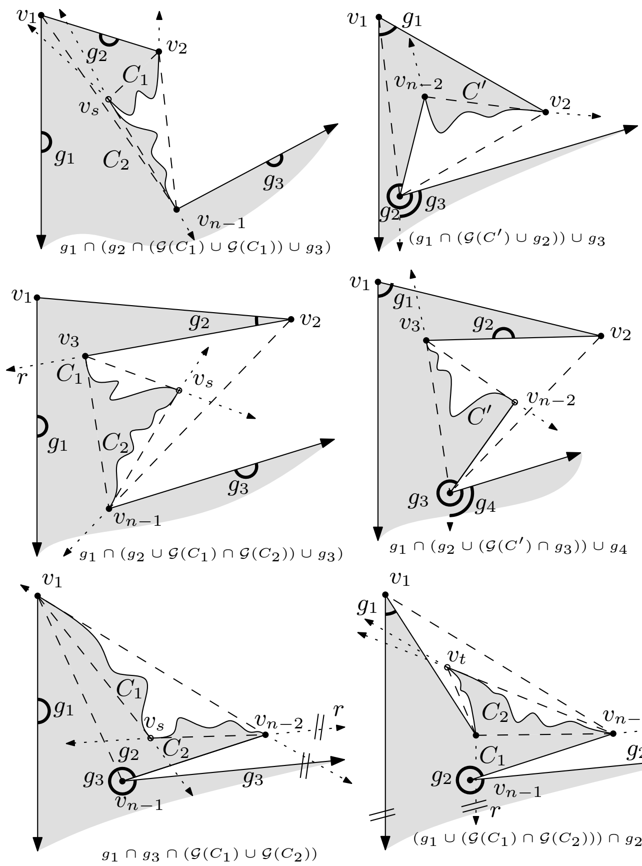

Figure 12: $S = \{v_1, v_2, v_{n-1}\}$ or $S = \{v_1, v_{n-2}, v_{n-1}\}$.

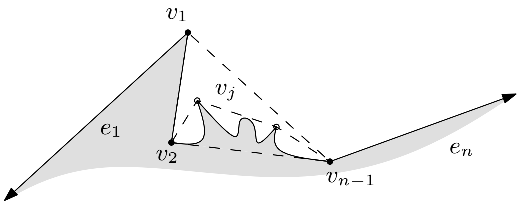

Figure 13: $S = \{v_1, v_2, v_{n-1}\}$.

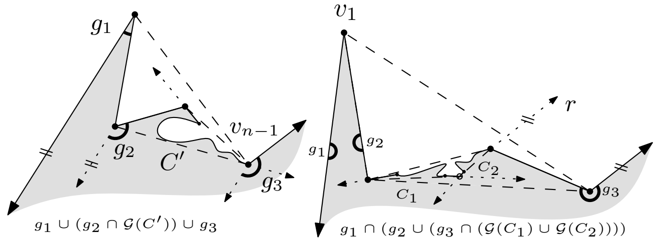

Figure 14: The only new vertex is v_3 or v_{n-1}.

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