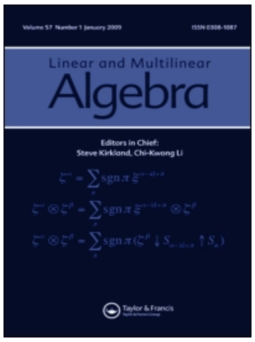

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On the uniqueness of the Z_1-eigenvector of transition probability tensors

J. Culp, K. Pearson and T. Zhang

Department of Mathematics & Statistics, Murray State University, Murray, KY, USA

ABSTRACT

Transition probability tensors of order 3 in dimension 3 and of order 4 in dimension 2 are studied. In both cases, we prove that an irreducible symmetric transition probability tensor has a unique positive Z_1-eigenvector.

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1. Introduction

Transition probability tensors and their associated Z_1-eigenvalues have been studied in the literature. [1–6] Of particular note, the uniqueness of the positive eigenvector corresponding to the unique Z_1-eigenvalue is in general unknown. Chang and Zhang [2] studied various techniques of contraction mapping, monotone operator method and index method. In [2], it was shown that an irreducible transition probability tensors of order 3 in dimension 2 have a unique Z_1-eigenvector corresponding to the unique Z_1-eigenvalue. In this paper, we focus on symmetric irreducible transition probability tensors of order 4 in dimension 2 and of order 3 in dimension 3. We are able to prove that a symmetric irreducible transition probability tensor in these orders and dimensions has a unique Z_1-eigenvector. The methods employed are direct, using Gröbner bases with the aid of CoCoA [7] and the computational software Mathematica. [8]

2. Notation

Let $\mathbb{R}$ be the real field; we consider an m-order n dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \quad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, \ldots, i_m \leq n.$$

A real m-order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is said to be nonnegative if each entry $a_{i_1 \cdots i_m} \geq 0$.

CONTACT K. Pearson kpearson@murraystate.edu

To an n-vector $x = (x_1, \ldots, x_n)$, real or complex, we define a n-vector:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n}.$$

We denote the set of all m-order n dimensional real tensors via $\mathbb{R}^{[m,n]}$ and the set of all m-order n dimensional nonnegative tensors via $\mathbb{R}_+^{[m,n]}$.

Definition 2.1: A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called *symmetric* if $a_{i_1\cdots i_m} = a_{\sigma(i_1\cdots i_m)}$ for all σ in the symmetric group on m indices.

The following definition of eigenvalues of tensors due to Qi [9] and Lim [10] have since become a standard terminology. The notion of the Z_1-eigenvalue in particular was introduced in [2].

Definition 2.2: Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonzero tensor. A pair $(\lambda, x) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ is called a Z_1-eigenvalue and Z_1-eigenvector (or simply Z_1-*eigenpair*) of $\mathcal{A}$ if they satisfy the equation

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ \sum_{i=1}^{n} x_i = 1. \end{cases} \tag{2}$$

The following notion of reducibility was introduced in [10,11].

Definition 2.3: A tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called *reducible*, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1\cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ *irreducible*.

The notion of transition probability tensor was introduced in [4]:

Definition 2.4: A nonnegative tensor $\mathcal{P} \in \mathbb{R}_+^{[m,n]}$ is called a transition probability tensor if $\mathcal{P} = (p_{i_1\cdots i_m})$ satisfies

$$\sum_{i=1}^{n} p_{ii_2\cdots i_m} = 1, \quad 1 \le i_2, \cdots, i_m \le n,$$

$$p_{i_1 i_2\cdots i_m} \ge 0, \quad 1 \le i_1, \cdots, i_m \le n.$$

3. Symmetric irreducible transition probability tensors in $\mathbb{R}_+^{[4,2]}$

In this section, we prove that a symmetric irreducible transition probability tensor in $\mathbb{R}_+^{[4,2]}$ has a unique Z_1-eigenvector. We note that an example was given in [2] of a positive tensor in $\mathbb{R}_+^{[4,2]}$ which lacked symmetry and had two Z_1-eigenvectors corresponding to a Z_1-eigenvalue of 1.

Theorem 3.1: *Let $\mathcal{P} \in \mathbb{R}_+^{[m,n]}$ be a symmetric transition probability tensor, then $x = (\frac{1}{n}, \cdots, \frac{1}{n})$ is a Z_1-eigenvector with a Z_1-eigenvalue of* 1.

Proof: Let $\mathcal{P} \in \mathbb{R}_+^{[m,n]}$ be a symmetric transition probability tensor, and let $x = (\frac{1}{n}, \cdots, \frac{1}{n})$. We have

$$(\mathcal{P}x^{m-1})_1 = \sum_{i_2 \cdots i_m = 1}^{n} p_{1 i_2 i_3 \cdots i_m} x_{i_2} \cdots x_{i_m}$$
$$= \sum_{i_2 \cdots i_m = 1}^{n} p_{1 i_2 i_3 \cdots i_m} \left(\frac{1}{n}\right)^{m-1}.$$

We now rearrange the terms of $p_{1 i_2 i_3 \cdots i_m}$ and using symmetry and the fact that $\mathcal{P}$ is a transition probability tensor, we arrive at our desired conclusion.

We consider each $p_{11 i_3 \cdots i_m}$. Since $\mathcal{P}$ is a transition probability tensor, we have $p_{11 i_3 \cdots i_m} + p_{21 i_3 \cdots i_m} + \cdots + p_{n1 i_3 \cdots i_m} = 1$. By symmetry, we note that $p_{k1 i_3 \cdots i_m} = p_{1k i_3 \cdots i_m}$ for each $1 < k \leq n$.

It is now merely a counting argument. There are n^{m-2} possibilities for $i_3 \cdots i_m$. There are n elements in the list of $p_{1k i_3 \cdots i_m}$ as $1 \leq k \leq n$. This gives n^{m-1} unique elements. Hence, $(\mathcal{P}x^{m-1})_1 = n^{m-2}(\frac{1}{n})^{m-1} = \frac{1}{n}$ as required.

A similar argument shows $(\mathcal{P}x^{m-1})_i = \frac{1}{n}$ for all $1 \leq i \leq n$. □

Theorem 3.2: *Let $\mathcal{P} \in \mathbb{R}_+^{[4,2]}$ be a symmetric irreducible transition probability tensor. Then $\mathcal{P}$ has a unique Z_1-eigenvector corresponding to* 1.

Proof: From the Theorem 3.1, we know $(\frac{1}{2}, \frac{1}{2})$ is a Z_1-eigenvector. Let $\mathcal{P} \in \mathbb{R}_+^{[4,2]}$ be a symmetric irreducible transition probability tensor, and let x be a Z_1-eigenvector. Then $\mathcal{P}$ and x must satisfy the following properties

$$\sum_{i_2, i_3, i_4 = 1}^{2} p_{i i_2 i_3 i_4} x_{i_2} x_{i_3} x_{i_4} = x_i, \ 1 \leq i \leq 2.$$
$$x_1 + x_2 = 1.$$

Given that $\mathcal{P}$ is a symmetric transition probability tensor, we make the following substitutions: We set $p_{1111} = a$, $p_{1112} = b$, $p_{1122} = c$, $p_{1222} = d$, and $p_{2222} = e$. For ease of notation, we also designate $x := x_1$ and $y := x_2$.

This gives

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = x$$
$$bx^3 + 3cx^2y + 3dxy^2 + ey^3 = y$$
$$x + y = 1$$

We now note that $a + b = 1$, $d + e = 1$, $b + c = 1$ and $c + d = 1$ since $\mathcal{P}$ is a transition probability tensor. This gives $a = c$ and $d = e$. This transforms our system into

$$ax^3 + 3(1-a)x^2y + 3axy^2 + (1-a)y^3 = x$$
$$(1-a)x^3 + 3ax^2y + 3(1-a)xy^2 + ay^3 = y$$
$$x + y = 1$$

By setting $y = 1 - x$ and solving the first equation above, we have $(2x - 1)((4a - 2)x^2 + (2 - 4a)x + a - 1) = 0$. We suppose $x \neq \frac{1}{2}$. Then $(4a - 2)x^2 + (2 - 4a)x + a - 1 = 0$. If $a = \frac{1}{2}$, then this equation becomes $-\frac{1}{2} = 0$, a contradiction. Hence, $2a - 1 \neq 0$. This implies $x = \frac{2a-1 \pm \sqrt{2a-1}}{2(2a-1)}$. Since $0 \leq a \leq 1$, we have $2a - 1 \leq 1$. Hence, $2a - 1 \leq \sqrt{2a - 1}$. We then have $\frac{2a-1-\sqrt{2a-1}}{2(2a-1)} < 0$ and $\frac{2a-1+\sqrt{2a-1}}{2(2a-1)} > 1$. This is a contradiction since $0 \leq x \leq 1$. □

4. Symmetric irreducible transition probability tensors in $\mathbb{R}_+^{[3,3]}$

In this section, we prove that a symmetric irreducible transition probability tensor in $\mathbb{R}_+^{[3,3]}$ has a unique Z_1-eigenvector.

Theorem 4.1: *Let $\mathcal{P} \in \mathbb{R}_+^{[3,3]}$ be an symmetric transition probability tensor, then $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a Z_1-eigenvector with a Z_1-eigenvalue of 1.*

Proof: This follows from Theorem 3.1. □

Let $\mathcal{P} \in \mathbb{R}_+^{[3,3]}$ be a symmetric irreducible transition probability tensor, and let x be a Z_1-eigenvector. Then $\mathcal{P}$ and x must satisfy the following properties

$$\sum_{i_2,i_3=1}^{3} p_{ii_2i_3} x_{i_2} x_{i_3} = x_i, \; 1 \leq i \leq 3.$$

$$\sum_{k=1}^{3} x_k = 1.$$

Given that $\mathcal{P}$ is a symmetric transition probability tensor, we make the following substitutions: We set $p_{111} = a$, $p_{112} = b$, $p_{122} = c$, and $p_{222} = d$. We have the following inequalities

(1) $p_{113} = p_{131} = p_{311} = 1 - p_{111} - p_{211} = 1 - a - b \geq 0$
(2) $p_{123} = p_{132} = p_{213} = p_{231} = p_{321} = p_{312} = 1 - p_{112} - p_{212} = 1 - b - c \geq 0$
(3) $p_{133} = p_{313} = p_{331} = 1 - p_{113} - p_{213} = a + 2b + c - 1 \geq 0$
(4) $p_{223} = p_{232} = p_{322} = 1 - p_{122} - p_{222} = 1 - c - d \geq 0$
(5) $p_{233} = p_{323} = p_{332} = 1 - p_{123} - p_{223} = b + 2c + d - 1 \geq 0$
(6) $p_{333} = 3(1 - b - c) - a - d \geq 0$

Using the above substitutions together with $x_3 = 1 - x_1 - x_2$, $x := x_1$, and $y := x_2$, we obtain the following system of polynomials to solve:

$$\begin{aligned} p &:= ax^2 + 2bxy + cy^2 + 2(1 - a - b)x(1 - x - y) + (a + 2b + c - 1)(1 - x - y)^2 \\ &\quad + 2(1 - b - c)y(1 - x - y) - x = 0 \\ q &:= bx^2 + 2cxy + dy^2 + 2(1 - b - c)x(1 - x - y) + (b + 2c + d - 1)(1 - x - y)^2 \\ &\quad + 2(1 - c - d)y(1 - x - y) - y = 0 \\ r &:= (1 - a - b)x^2 + 2(1 - b - c)xy + (1 - c - d)y^2 + 2(a + 2b + c - 1)x(1 - x - y) \\ &\quad + (3(1 - b - c) - a - d)(1 - x - y)^2 + 2(b + 2c + d - 1)y(1 - x - y) \\ &\quad - (1 - x - y) = 0 \end{aligned}$$

Lemma 4.2: *Let $\mathcal{P} \in \mathbb{R}_+^{[3,3]}$ be a symmetric irreducible transition probability tensor. Suppose x is a Z_1-eigenvector with $x_2 = \frac{1}{3}$. Then $x_1 = x_2 = x_3 = \frac{1}{3}$.*

Proof: Using CoCoA [7], we form the ideal generated by the polynomials p, q, r together with the polynomial $3x_2 - 1$ in the ring $\mathbb{Q}[a, b, c, d, x_1, x_2]$. We obtain the second element of its Gröbner basis is $(3x_1 - 1)(4b + 4c + d - 3)$ and the third element is $(3x_1 - 1)(12ax_1 - 9cx_1 - 3dx_1 - 4a + 3c + d - 3)$. If $x_1 \neq \frac{1}{3}$, then both $4b + 4c + d - 3 = 0$ and $12ax_1 - 9cx_1 - 3dx_1 - 4a + 3c + d - 3 = 0$. Using Mathematica [8], there is no solution to this system of equations when considering the inequalities (1)–(6). Hence, we must have that $x_1 = \frac{1}{3}$ as required. □

Theorem 4.3: *Let $\mathcal{P} \in \mathbb{R}_+^{[3,3]}$ be a symmetric irreducible transition probability tensor. Then $\mathcal{P}$ has a unique Z_1-eigenvector corresponding to 1.*

Proof: Using CoCoA [7], we form the ideal generated by the polynomials p, q, r and consider its Gröbner basis. The fourth element of the Gröbner basis is:

$$\begin{aligned}&(3x_2 - 1)(-6bcx_1x_2 - 6c^2x_1x_2 + 6adx_1x_2 + 6bdx_1x_2 - 3bcx_2^2 - 12c^2x_2^2 + 3adx_2^2 + 12bdx_2^2\\&+ 2bcx_1 + 2c^2x_1 - 2adx_1 - 2bdx_1 + 4bcx_2 + 10c^2x_2 - 4adx_2 - 10bdx_2 - 2ax_1x_2 + 6cx_1x_2\\&- 4dx_1x_2 - ax_2^2 - 3bx_2^2 + 9cx_2^2 - 5dx_2^2 - bc - 2c^2 + ad + 2bd - 2ax_1 - 4bx_1 - 4cx_1\\&+ dx_1 - 4bx_2 - 13cx_2 + 2dx_2 + a + 3b + 4c + 3x_1 + 5x_2 - 2)\end{aligned}$$

Define the polynomial f to be the second factor above. If $3x_2 - 1 = 0$, then by the previous lemma, we are done. We assume $3x_2 - 1 \neq 0$; this implies $f = 0$. We form the ideal generated by p, q, r, f and eliminate x_1 to obtain a polynomial cubic in x_2:

$$\begin{aligned}g := {}& 81b^2c^2x_2^3 - 108ac^3x_2^3 - 108b^3dx_2^3 + 162abcdx_2^3 - 27a^2d^2x_2^3 - 81b^2c^2x_2^2\\&+ 108ac^3x_2^2 + 108b^3dx_2^2 - 162abcdx_2^2 + 27a^2d^2x_2^2 + 36b^3x_2^3 - 54abcx_2^3\\&- 54b^2cx_2^3 + 108ac^2x_2^3 - 54bc^2x_2^3 + 36c^3x_2^3 + 18a^2dx_2^3 - 54abdx_2^3 + 108b^2dx_2^3\\&- 54acdx_2^3 - 54bcdx_2^3 + 18ad^2x_2^3 + 27b^2c^2x_2 - 36ac^3x_2 - 36b^3dx_2 + 54abcdx_2\\&- 9a^2d^2x_2 + 12b^3x_2^2 - 18abcx_2^2 + 90b^2cx_2^2 - 180ac^2x_2^2 + 18bc^2x_2^2 - 60c^3x_2^2\\&+ 6a^2dx_2^2 + 90abdx_2^2 - 36b^2dx_2^2 + 18acdx_2^2 + 90bcdx_2^2 - 30ad^2x_2^2 - 3a^2x_2^3\\&+ 18abx_2^3 - 27b^2x_2^3 - 18acx_2^3 + 54bcx_2^3 - 27c^2x_2^3 + 6adx_2^3 - 18bdx_2^3 + 18cdx_2^3\\&- 3d^2x_2^3 - 3b^2c^2 + 4ac^3 + 4b^3d - 6abcd + a^2d^2 - 20b^3x_2 + 30abcx_2 - 42b^2cx_2\\&+ 84ac^2x_2 + 6bc^2x_2 + 28c^3x_2 - 10a^2dx_2 - 42abdx_2 - 12b^2dx_2 + 6acdx_2\\&- 42bcdx_2 + 14ad^2x_2 - 5a^2x_2^2 - 6abx_2^2 - 57b^2x_2^2 + 102acx_2^2 - 42bcx_2^2 + 87c^2x_2^2\\&- 14adx_2^2 - 54bdx_2^2 - 18cdx_2^2 + 7d^2x_2^2 + 4b^3 - 6abc + 6b^2c - 12ac^2 - 2bc^2 - 4c^3\\&+ 2a^2d + 6abd + 4b^2d - 2acd + 6bcd - 2ad^2 - a^2x_2 - 18abx_2 + 19b^2x_2 - 62acx_2\\&- 22bcx_2 - 62c^2x_2 + 10adx_2 + 35bdx_2 - 2cdx_2 - 5d^2x_2 - 12ax_2^2 + 36bx_2^2 - 36cx_2^2\\&+ 12dx_2^2 + a^2 + 6ab + b^2 + 10ac + 10bc + 12c^2 - 2ad - 5bd + 2cd + d^2 + 16ax_2\\&+ 13bx_2 + 46cx_2 - 3dx_2 - 4a - 7b - 10c - 12x_2 + 3\end{aligned}$$

We use Mathematica [8] to solve for the $f = 0$ and $g = 0$ together with the inequalities (1)–(6) and see there is no solution. □

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No potential conflict of interest was reported by the authors.

References

[1] Chang KC, Pearson K, Zhang T. Some variational principles of the Z-eigenvalues for nonnegative tensors. Linear Algebra Appl. 2013;438:4166–4182.
[2] Chang KC, Zhang T. On the uniqueness and non-uniqueness of the positive Z-eigenvector for transition probability tensors. J. Math. Anal. Appl. 2013;408:525–540.
[3] Hu S, Qi L. Convergence of a second order Markov chain. Appl. Math. Comput. 2014;241: 183–192.
[4] Li W, Ng M. Existence and uniqueness of stationary probability vector of a transition probability tensor. Hong Kong: Department of Mathematics, The Hong Kong Baptist University; 2011.
[5] Li W, Ng M. On linear convergence of power method for computing stationary probability vector of a transition probability tensor. Hong Kong: Department of Mathematics, The Hong Kong Baptist University; 2011.
[6] Li W, Ng M. On the limiting probability distribution of a transition probability tensor. Linear Multilinear Algebra. 2014;62:362–385.
[7] Abbott J, Bigatti AM, Lagorio G. CoCoA-5: a system for doing Computations in Commutative Algebra; 2015. Available from: http://cocoa.dima.unige.it.
[8] Wolfram Research, Inc. Mathematica 8.0. 2010. Available from: https://www.wolfram.com.
[9] Qi L. Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. 2005;40:1302–1324.
[10] Lim LH. Singular values and eigenvalues of tensors, A variational approach. In: Proceeding 1st IEEE International workshop on computational advances of multi-tensor adaptive processing; 2005 Dec 13–15; Puerto Vallarto, Mexico. p. 129–132.
[11] Chang KC, Pearson K, Zhang T. Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. 2008;6:507–520.