

Degree sum conditions in graph pebbling

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Abstract

Given a graph G on n vertices and a distribution, D , of pebbles on the vertices of G , we define a *pebbling move* to be the removal of two pebbles from a given vertex and the placement of one on an adjacent vertex. If D has n pebbles and if after a sequence of pebbling moves we can place a pebble on any specified vertex then we call G Class 0. We give a sufficient degree sum condition for G to be Class 0.

Keywords: pebbling, pebbling number, degree sum condition

1 Introduction

We let $G = (V, E)$ be a simple graph with vertex set $V := V(G)$ and edge set $E := E(G)$ where $|V| = n$. For sets $A, B \subset V$ and $A \cap B = \emptyset$, we use $G[A, B]$ to denote the bipartite subgraph of G containing all edges with one end-vertex in each of A and B . We define the degree of a vertex v , denoted $d(v)$, to be the number of edges incident with v and denote its set of neighbors by $N(v)$. The minimum degree, maximum degree and independence number of a graph G are denoted $\delta(G)$, $\Delta(G)$ and $\alpha(G)$, respectively. We let $\sigma_k(G) = \min\{d(x_1) + \dots + d(x_k) \mid x_1, \dots, x_k \text{ are independent in } G\}$.

Given a distribution D of pebbles on the vertices of G , which may be thought of as an assignment of integer weights to the vertices of G , we say that a *pebbling move* consists of removing two pebbles from a vertex and then placing one pebble on an adjacent vertex. The number of pebbles placed on a vertex v is denoted $D(v)$. Given a target vertex r , known as the *root vertex*, we say that r can be *reached*, or *pebbled*, if after a sequence of pebbling moves it is possible to place a

pebble on r . The *pebbling number* of G , $\pi(G)$, is the least integer m such that, regardless of how m pebbles are distributed on the vertices of G , after a sequence of pebbling moves we can reach any vertex. It is easy to see that $\pi(G) > n - 1$ since placing each of $n - 1$ pebbles on a distinct vertex leaves one vertex, r , without a pebble and no pebbling moves possible. Graphs for which $\pi(G) = n$ are known as Class 0 graphs and this class is the object of our consideration. It is obvious that such graphs must be connected and in fact must be 2-connected. The latter is seen true if we let x be a cut-vertex of G , let the components of $G(V \setminus \{x\}, E)$ be G_1, G_2 and $v_i \in G_i$ and consider the following distribution in G of n pebbles, $D(v_1) = 3, D(v_2) = 0, D(x) = 0$ and $D(v) = 1$ for all other vertices. The distribution does not allow v_2 to be pebbled.

In [4], the problem of determining necessary and sufficient conditions for a graph G to be Class 0 is given. Most results in this direction, including those surveyed in [4], focus on conditions on the diameter and connectivity of G . A result in [5], which we discuss below, gives a sufficient condition in regards to the number of edges of G . Here, we give a sufficient degree sum condition, which is best possible, for G to be Class 0.

Theorem 1 *If $\sigma_2(G) \geq n$, then G is Class 0.*

The proof of Theorem 1 is essentially the same as the proof of Theorem 2 in Czygrinow and Hurlbert [2], so we do not present it here. However, as a result we obtain the following.

Corollary 2 *If $\delta(G) \geq \lceil \frac{n}{2} \rceil$, then G is Class 0.*

In [2], it was incorrectly claimed that if $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$ then G is Class 0. The error in [2] occurred in the proof of the lower bound on $\delta(G)$. To see this, consider when n is odd the following graph G - which has minimum degree $\lfloor \frac{n}{2} \rfloor$, but is not Class 0. Let G be the graph of two complete graphs of order $\lceil \frac{n}{2} \rceil$ intersecting in a single vertex. This graph contains a cut-vertex, so it cannot be Class 0. Thus we must necessarily have $\delta(G) \geq \lceil \frac{n}{2} \rceil$. The proof that this bound is sufficient to guarantee membership in Class 0 holds as given in [2] with $\lfloor \frac{n}{2} \rfloor$ replaced by $\lceil \frac{n}{2} \rceil$, but also follows immediately from Theorem 1.

We are able to offer the following new result.

Theorem 3 *Let G be a graph on $n \geq 6$ vertices. If for each maximal independent set, S , of G we have*

$$\sum_{v \in S} d(v) \geq (|S| - 1)(n - |S|) + 2$$

then G is Class 0.

Using this result, we can see that the complete multipartite-graph, K_{p_1, \dots, p_t} , with partite sets P_1, \dots, P_t where $|P_i| = p_i$, $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \dots \leq p_t$, is Class 0 as long as $\sum p_i \geq 6$, except in the case $t = 2$ and $p_1 = 1$. Notice that the only maximal independent sets are P_1, \dots, P_t .

We also obtain the following corollary due to Pachter, Snevily and Voxman [5].

Corollary 4 [5] *If G is a graph on $n \geq 4$ vertices and $|E(G)| \geq \binom{n-1}{2} + 2$, then G is Class 0.*

For a survey of results in graph pebbling, we refer the reader to [3] and [4].

2 Proof of Main Result

To see that the conditions given in Theorem 1 and Theorem 3 are best possible consider the following construction. Let G' be any graph on $n - k$ vertices with vertex set $\{x_1, \dots, x_{n-k}\}$. To G' we add vertices $\{x_{n-k+1}, \dots, x_n\}$ with each vertex in $\{x_{n-k+1}, \dots, x_{n-1}\}$ adjacent to each vertex in $\{x_1, \dots, x_{n-k}\}$ and x_n adjacent only to x_1 , denote this graph by G . In G , the set of vertices x_{n-k+1}, \dots, x_n forms an independent set of size k with degree sum $(k - 1)(n - k) + 1$, yet G is not Class 0 as it contains a cut-vertex, x_1 .

To prove Theorem 3 we first present a result from [1]. To do this, we must first describe a class of graphs \mathcal{F} . The class, \mathcal{F} , we give is a correction to the one given in [1], yet the corresponding result (and its proof) still holds.

We refer the reader to Figure 1. To begin, each $F \in \mathcal{F}$ has a six-cycle $C_6 = pr'qp'rqp$. For each vertex p, q and r there exists a subgraph (possibly the empty graph), denoted H_p, H_q and H_r , respectively. In each component of H_p there exists a vertex adjacent to p (denoted by double dotted lines), and each vertex in H_p is adjacent to q', r' (denoted by solid lines). Similar statements hold for H_q and H_r . Further, in each F there exists a subgraph, denoted H_c , in which each vertex is adjacent to at least two of $\{p', q', r'\}$ (denoted by arrowed lines). At least two of the edges $p'q', q'r', r'p'$ exist (denoted by dashed lines). Edges may not exist between any pair H_i, H_j . This describes all possible edges of a member of \mathcal{F} .

Theorem 5 [1] *If a graph G on $n \geq 6$ vertices has diameter two, connectivity at least two and $\pi(G) > n$, then $G \in \mathcal{F}$.*

The class of graphs given in [1] differs from the class \mathcal{F} . It differs from \mathcal{F} only in H_c , all other aspects are the same. In [1], H_c was essentially described as consisting of two parts $H_{c'}$ and $H_{c''}$, where $H_{c'}$ consists of those vertices in H_c adjacent to vertices p' and q' only and $H_{c''}$ consists of

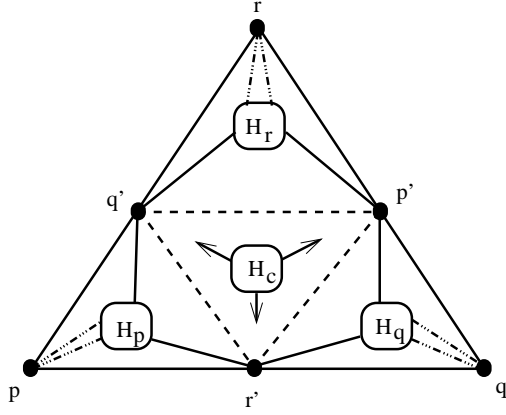


Figure 1: The family \mathcal{F}

all other vertices in H_c . It was specified that edges did not exist between $H_{c'}$ and $H_{c''}$, however we show that this need not be the case and this leads to our description of \mathcal{F} as given above.

To see that the description of \mathcal{F} given here is more broad than the one given in [1], we give a graph F which has diameter two and connectivity two, is not Class 0 and is not a member of the family of graphs described in [1]. We refer the reader to the graph given in Figure 2. By inspection, it is easy to see that F has diameter two and connectivity two. The graph F is not Class 0 since if we let $D(p) = D(q) = 3, D(r) = D(p') = D(q') = D(r') = 0$ and $D(x) = D(y) = D(z) = 1$ then it is impossible to pebble r .

A consequence of Theorem 5, as shown in [1], is that if a graph G has diameter two and connectivity at least three then G is Class 0. The discussion following the statement of Theorem 3 gives an instance of a graph shown to be Class 0 by Theorem 3, but not by this consequence - namely the graph $K_{2,n-2}$.

We now give two preparatory propositions towards the proof of Theorem 3. Given a graph $F \in \mathcal{F}$ and a set of vertices, R , in $V(F)$ we say that R has *Property 1* if R contains a vertex in each of $H_p \cup p, H_q \cup q$ and $H_c \cup H_r \cup r$ and does not contain any element of $\{p', q', r'\}$.

Proposition 6 *If $F \in \mathcal{F}$ and $\alpha(F) \geq 3$, then for each $s, 3 \leq s \leq \alpha(F)$, there is an independent set of size s in F with Property 1.*

PROOF: Let $F \in \mathcal{F}$ and S be an independent set in $V(F)$ of size at least three. Assume that S does not have Property 1 and we will show that there exists an independent set T with $|T| = |S|$ having Property 1.

If S does not contain any vertex from $\{p', q', r'\}$, then as there are no edges between $H_p \cup p$ and $H_q \cup q, H_p \cup p$ and $H_c \cup H_r \cup r$, and $H_q \cup q$ and $H_c \cup H_r \cup r$, we may remove a single vertex from

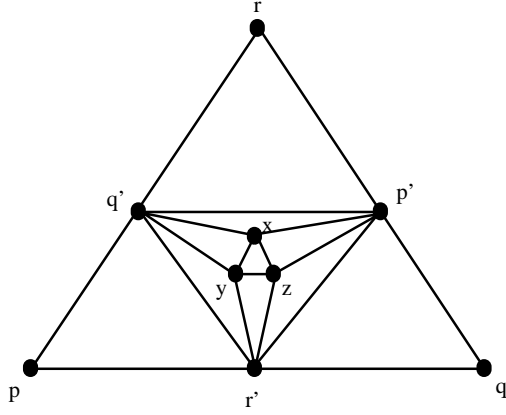


Figure 2: A graph F with diameter two, connectivity two, but not Class 0

S and replace it by any vertex from the set which it does not intersect, while at the same time ensuring the number of sets it intersects increases, to form S' . If S' has Property 1 then we let $T = S'$. Otherwise, we repeat this procedure to form S'' , at which point we are guaranteed that S'' has Property 1 and so we let $T = S''$.

Otherwise, S does contain a vertex from $\{p', q', r'\}$. The set S may contain at most two vertices from $\{p', q', r'\}$, as this set induces at least two edges. First consider if S contains precisely one such vertex. Let's say $p' \in S$, then as p' is adjacent to each vertex in $H_q \cup q$ and $r' \notin S$ we may replace p' by any vertex in $H_q \cup q$ to form an independent set, S' . Similarly, if $q' \in S$, or $r' \in S$, then a similar procedure may be performed to form an independent set, S' . The set S' has $|S'| = |S|$ and does not contain any vertex in $\{p', q', r'\}$, so by the previous case either S' has Property 1 or we may find a suitable set T .

Finally, we consider the case in which S contains two vertices from $\{p', q', r'\}$. Regardless of the choice of the two, every remaining vertex in the graph will be adjacent to at least one of the two. Thus the set S cannot contain any other vertices from G . This contradicts that the size of S is at least three. \square

For a positive integer a , we define a positive integer partition of length t of a to be a vector $\mathbf{a} = (a_1, \dots, a_t)$ such that $a_1 + \dots + a_t = a$ and for $1 \leq i \leq t$ we have $a_i \in \mathbb{Z}^+$.

Proposition 7 *Let a, b, t be positive integers with $b \geq a \geq t$. Let \mathbf{a}, \mathbf{b} be positive integer partitions of length t of a and b , respectively. If for $1 \leq i \leq t$ we have $b_i \geq a_i$ then*

$$f(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^t a_i(b_i - a_i) \tag{1}$$

is maximized when for some i we have $a_i = a - (t - 1), b_i = b - (t - 1)$, and so $a_j = b_j = 1$ for

all $j \neq i$ and $f(\mathbf{a}, \mathbf{b}) = (a - (t - 1))(b - a)$.

We delay the proof of Proposition 7, a purely number theoretic result, until after the proof of our main result which we now give.

The proof of Theorem 3 is based on arguments given in [2].

PROOF OF THEOREM 3: Let G be given according to the conditions of Theorem 3. If $\alpha(G) = 1$, then $G = K_n$ and the result holds trivially. If $\alpha(G) = 2$, then the condition of Theorem 1 holds and G is Class 0.

Thus we may assume that $\alpha(G) \geq 3$ and suppose G is not Class 0. We begin by showing that G must belong to \mathcal{F} . Let $x, y \in V(G)$ such that $xy \notin E(G)$ and let S be any maximal independent set containing both x and y . As S is independent, the maximum degree of a vertex in S is $n - |S|$. This fact and the hypothesis imply that we have,

$$\begin{aligned} d(x) + d(y) &\geq (|S| - 1)(n - |S|) + 2 - (|S| - 2)(n - |S|) \\ &= (n - |S|) + 2. \end{aligned}$$

The pigeonhole principle implies that x and y must share at least two common neighbors, and so the diameter of G is at most two. The diameter is at least two since $\alpha(G) \geq 3$, and so the diameter must be equal to two. We can also reach the conclusion that between any pair of non-adjacent vertices there exists at least two vertex disjoint x, y -paths. Now consider $x, y \in V(G)$ such that $xy \in E(G)$, we seek to find an x, y -path distinct from the edge xy . This will show that between any two vertices in G there exists two vertex disjoint paths and so, by a theorem of Whitney [6], G is 2-connected. As G is connected at least one of x and y has another neighbor, say x does and call x 's neighbor u . If $uy \in E(G)$ then uy is the second path we seek. Thus we may assume that $uy \notin E(G)$ and let S' be a maximal independent set containing both u and y . Then, as above, we may show that u and y have at least two neighbors in common, one of which, say v , is distinct from x . We then have $xuvy$ as an x, y -path distinct from the edge xy . Thus G is 2-connected.

As G has diameter 2, is 2-connected and, by assumption, is not Class 0, then by Theorem 5 G is in \mathcal{F} . Now consider a maximal independent set S such that $|S| = \alpha(G)$. We apply Proposition 6 to S to obtain an independent set T with $|T| = \alpha(G)$ and T has Property 1. Let's say that i vertices from T are in $H_p \cup p$, j vertices from T are in $H_q \cup q$ and k vertices from T are in $H_c \cup H_r \cup r$. We then have the following,

$$\begin{aligned}
\sum_{u \in T, u \in H_p \cup p} d(u) + \sum_{v \in T, v \in H_q \cup q} d(v) + \sum_{w \in T, w \in H_c \cup H_r \cup r} d(w) &\leq i(|H_p \cup p| + 2 - i) + j(|H_q \cup q| + 2 - j) \\
&\quad + k(|H_c \cup H_r \cup r| + 2 - k) \\
&= 2(i + j + k) + i(|H_p \cup p| - i) + j(|H_q \cup q| - j) \\
&\quad + k(|H_c \cup H_r \cup r| - k) \\
&= 2\alpha(G) + i(|H_p \cup p| - i) + j(|H_q \cup q| - j) \\
&\quad + k(|H_c \cup H_r \cup r| - k).
\end{aligned}$$

Note that $i, j, k \geq 1$, $i + j + k = \alpha(G)$ and $|H_p \cup p| + |H_q \cup q| + |H_r + H_c + r| = n - 3$. We may now apply Proposition 7 with $a = \alpha(G)$, $b = (n - 3)$ and $t = 3$. As a result, the sum on the right-hand side of the above inequality is at most $2\alpha(G) + (\alpha(G) - 2)(n - 3 - \alpha(G))$. However, this quantity is less than $(\alpha(G) - 1)(n - \alpha(G)) + 2$ when $n > 4$. That is, we obtain a contradiction to the degree sum condition. Thus G is Class 0. \square

We now present the proof of Corollary 4.

PROOF OF COROLLARY 4: For $n = 4, 5$, we can see by inspection that the claim is true. Now, for $n \geq 6$ let $G = (V, E)$ be as given and consider any independent set S in G . By the edge count we see that G has at most $n - 3$ non-edges. The set S contains exactly $\binom{|S|}{2}$ non-edges and so $G[S, V \setminus S]$ contains at most $n - 3 - \binom{|S|}{2}$ non-edges. We then have,

$$\begin{aligned}
\sum_{v \in S} d(v) &\geq |S|(n - |S|) - (n - 3 - \binom{|S|}{2}) \\
&\geq (|S| - 1)(n - |S|) + 2.
\end{aligned}$$

Thus, by Theorem 3, G is Class 0. \square

We now present the proof of Proposition 7.

PROOF OF PROPOSITION 7: Let a, b, t be positive integers with $b \geq a \geq t$ and let \mathbf{a}, \mathbf{b} be any positive integer partitions of length t , respectively, for which $b_i \geq a_i$, $1 \leq i \leq t$.

First suppose that $a = b$. In this case, $\sum a_i = \sum b_i$ and so $\sum(b_i - a_i) = 0$. As $b_i \geq a_i$ we must have that $b_i = a_i$ for all i . Thus $f(\mathbf{a}, \mathbf{b}) = 0$ and the conclusion holds true trivially.

We may now consider when $b > a$. As $b > a$, there is an i for which $b_i > a_i$. Let $d_i = b_i - a_i > 0$. If for some $j \neq i$ we have $a_j \geq a_i$ then choose the largest such a_j . If there is more than one choice, then of these choose the one with the largest such j . We may then replace \mathbf{b} by $\mathbf{b}_1 = (b_1, \dots, b_i - d_i, \dots, b_j + d_i, \dots, b_t)$. We then have that $f(\mathbf{a}, \mathbf{b}_1) \geq f(\mathbf{a}, \mathbf{b})$ since

$$\begin{aligned}
f(\mathbf{a}, \mathbf{b}_1) - f(\mathbf{a}, \mathbf{b}) &= a_i(b_i - d_i - a_i) + a_j(b_j - a_j + d_i) - [a_i(b_i - a_i) + a_j(b_j - a_j)] \\
&= d_i(a_j - a_i) \geq 0, \quad \text{since } a_j \geq a_i.
\end{aligned}$$

Repeating this procedure until it is no longer possible allows us to replace \mathbf{b} by some \mathbf{b}_2 , so that in \mathbf{b}_2 there exists a unique j for which $b_j > a_j$. Fix this j .

In \mathbf{b}_2 for $i \neq j$ we have $a_i = b_i$. If we have for some i , $a_i, b_i > 1$ then we perform the following operation. Replace \mathbf{a} and \mathbf{b}_2 by $\mathbf{a}_3 = (a_1, \dots, a_i - (a_i - 1), \dots, a_j + (a_i - 1), \dots, a_t)$ and $\mathbf{b}_3 = (b_1, \dots, b_i - (b_i - 1), \dots, b_j + (b_i - 1), \dots, b_t)$, respectively. We then have that $f(\mathbf{a}_3, \mathbf{b}_3) > f(\mathbf{a}, \mathbf{b}_2)$ since,

$$\begin{aligned}
f(\mathbf{a}_3, \mathbf{b}_3) - f(\mathbf{a}, \mathbf{b}_2) &= (a_j + (a_i - 1))(b_j - a_j) - a_j(b_j - a_j) \\
&= (a_i - 1)(b_j - a_j) \\
&> 0.
\end{aligned}$$

Repeating this procedure until it is no longer possible allows us to replace \mathbf{a}, \mathbf{b}_2 by some $\mathbf{a}^*, \mathbf{b}^*$ so that there exists a unique j for which $b_j > a_j$ and for all $i \neq j$ we have $a_i = b_i = 1$ and $f(\mathbf{a}^*, \mathbf{b}^*) \geq f(\mathbf{a}, \mathbf{b})$. The conclusion now readily holds. \square

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