Pricing Network Edges for Heterogeneous Selfish Users

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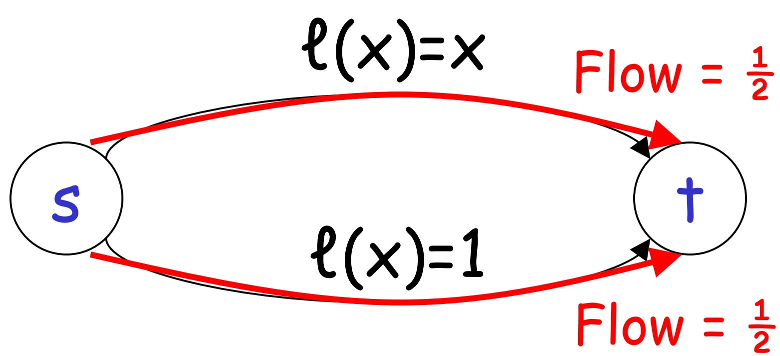
Selfish Routing

- a directed graph $G = (V,E)$
- a source s and a destination t
- one unit of traffic from s to t
- for each edge e, a latency function $\ell_e(\cdot)$
  - assumed continuous, nondecreasing, convex

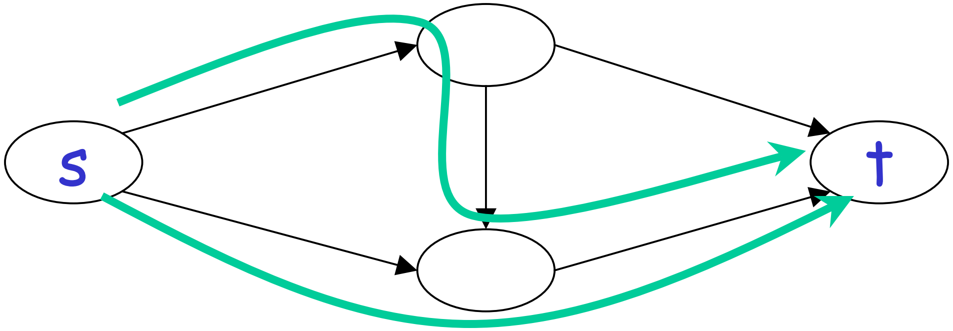
Example:



Routings of Traffic

Traffic and Flows:

- f_P = fraction of traffic routed on s-t path P
- flow vector **f** ⇔ routing of traffic



Selfish routing: what flows arise as the routes chosen by many noncooperative agents?

Nash Flows

Some assumptions:

- agents small relative to network
- want to minimize personal latency

Def: A flow is at Nash equilibrium (or is a Nash flow) if all flow is routed on min-latency paths [given current edge congestion]

- have existence, uniqueness [Wardrop, Beckmann et al 50s]

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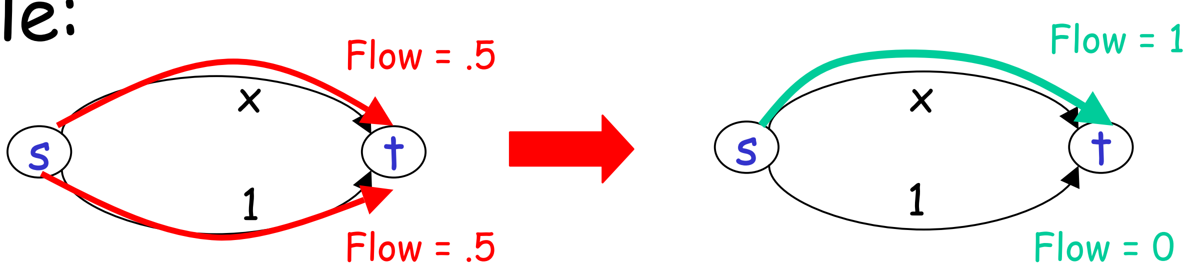
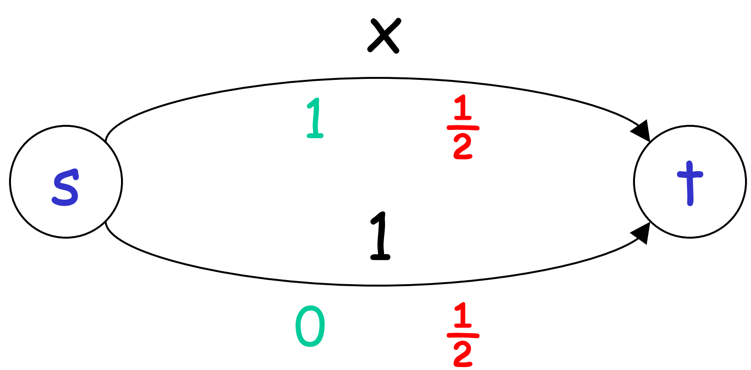


Inefficiency of Nash Flows

Our objective function: average latency

- $\Rightarrow$ Nash flows need not be optimal
- observed informally by [Pigou 1920]



- Average latency of Nash flow = $1 \cdot 1 + 0 \cdot 1 = 1$
- of optimal flow = $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$

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Thm: [folklore] marginal cost taxes w.r.t. the opt flow induce the opt flow as a Nash eq.

Why Homogeneous?

Problem: strong homogeneity assumption

- at odds with assumption of many users
- are taxes still powerful without this?

Our assumption: agent a has objective function $\text{time} + \beta(a) \times \text{money}$

- distribution function β assumed known
  - in aggregate sense

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Pf Idea: Brouwer's fixed-point thm.

- continuous map on a compact set has fixed pt
- want OPT-inducing taxes ⇔ fixed points

- continuous map:
  - given tax vector not inducing OPT, push vector in helpful direction (else fixed pt)

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  - do all fixed pts now give OPT-inducing taxes?

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- Key Lemma: for sufficiently large bound, yes!
  - requires nontrivial proof (cf., Braess's Paradox)

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- in fact, set of all such taxes described by poly-sized list of linear inequalities
  - based on [Bergendorff et al 97]
  - can optimize secondary linear objective
- existence thm $\Rightarrow$ there is a feasible point
  - otherwise set might be empty

When Taxes Cause Disutility

Observation: so far, min delay is holy grail; exorbitant taxes ignored

Question: are small taxes and min latency both possible?

- see EC '03 paper for many other questions
- see also "frugal mechanisms" [Archer/Tardos]

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Thm: precise characterization of distribution functions β where both are always possible.

- strong condition, satisfied only with many misers

My Favorite Open Question

Question: what remains true in multicommodity flow networks?

Note: Existence and algorithmic theorems will hold if truncation trick still works.

- need "key lemma" that no bad fixed points exist