Selfish Routing

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Traffic in Congested Networks

The Model:

- A directed graph $G = (V,E)$
- A source $s$ and a sink $t$
- A rate $r$ of traffic from $s$ to $t$
- For each edge $e$, a latency function $l_e(\cdot)$

Example: $(r=1)$
Flows and their Cost

Traffic and Flows:
• \( f_p \) = amount of traffic routed on s-t path \( P \)
• flow vector \( f \) ⇔ traffic pattern at steady-state

The Cost of a Flow:
• \( l_P(f) \) = sum of latencies of edges on \( P \) (w.r.t. the flow \( f \))
• \( C(f) \) = cost or total latency of flow \( f \):
  \[ \sum_P f_p \cdot l_P(f) \]
Flows and Game Theory

• flow = routes of many noncooperative agents

• Examples:
  - cars in a highway system
  - packets in a network
    • [at steady-state]

• cost (total latency) of a flow as a measure of social welfare

• agents are selfish
  - do not care about social welfare
  - want to minimize personal latency
Flows at Nash Equilibrium

**Def:** A flow is at Nash equilibrium (is a Nash flow) if no agent can improve its latency by changing its path.

Assumption: edge latency functions are continuous, nondecreasing.

**Lemma:** \( f \) is a Nash flow if and only if all flow travels along minimum-latency paths (w.r.t. \( f \)).
Nash Flows and Social Welfare

Central Question:

- What is the cost of the lack of coordination in a Nash flow?

\[
\begin{array}{c}
S \\
\times \\
1 \\
\frac{1}{2} \\
1 \\
0 \\
\frac{1}{2} \\
\uparrow
\end{array}
\]

- Cost of Nash = 1
- min-cost
  \[= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}\]

Analogous to IP versus ATM:

- ATM ≈ central control ≈ min cost
- IP ≈ no central control ≈ selfish
Previous Work

• [Beckmann et al. 56], …
  - Existence, uniqueness of flows at Nash equilibrium

• [Dafermos/Sparrow 69], …
  - Efficiently computing Nash and optimal flows

• [Braess 68], …
  - Network design

• [Koutsoupias/Papadimitriou 99]
  - Quantifying the cost of a lack of coordination
Braess’s Paradox

Better network, worse Nash flow:

Cost of Nash flow = 1.5

Cost of Nash flow = 2

All traffic experiences increased latency!
Our Results for Linear Latency

**Def:** a linear latency function is of the form $l_e(x) = a_e x + b_e$

**Theorem 1:**
linear latency fns $\Rightarrow$

\[
\text{cost of Nash flow} = \frac{4}{3} \times \text{cost of opt flow}
\]
General Latency Functions?

Bad Example: \( (r = 1, k \text{ large}) \)

\[
\begin{array}{c}
\text{s} \\
\uparrow \\
\times^k \\
\downarrow \\
\text{t}
\end{array}
\]

\( 1 \quad 1 - ? \)

\( 1 \)

\( 0 \quad ? \)

Nash flow has cost 1, \( \min \text{ cost} \approx 0 \)

\( \Rightarrow \) Nash flow can cost arbitrarily more than the optimal (min-cost) flow

- even if latency functions are polynomials
Our Results for General Latency

All is not lost: the previous example does not preclude interesting bicriteria results.

Theorem 2:
continuous, nondecreasing latency functions $\Rightarrow$

\[
\text{cost of } \text{Nash flow at rate } r = \text{cost of } \text{opt flow at rate } 2r
\]
Optimal Flows + Convexity

Minimize

\[ C(f) = \sum_e f_e \cdot l_e(f_e) \]

- by summing over edges rather than paths
- \( f_e \) amount of flow on edge \( e \)

Cost \( C(f) \) usually convex
- e.g., if \( l_e(f_e) \) convex
- if \( l_e(f_e) = a_e f_e + b_e \)

\[ \Rightarrow \quad C(f) = \sum_e f_e \cdot (a_e f_e + b_e) \]
(\text{convex quadratic})
Why Is Convexity Good?

A solution is optimal for a convex fn if and only if
- tiny change in a locally feasible direction cannot decrease the cost
Characterizing the Optimal Flow

Direction of change: moving a small amount of flow from one path to another

flow $f$ is minimum cost iff its cost cannot be improved in this way
Characterizing the Optimal Flow

Cost $f_e \cdot l_e(f_e) \Rightarrow$ marginal cost of increasing flow on edge $e$ is

$$l_e(f_e) + f_e \cdot l_e'(f_e)$$

latency of new flow

Added latency of flow already on edge

**Key Lemma:** a flow $f$ is optimal if and only if all flow travels along paths with minimum marginal cost (w.r.t. $f$).
The Optimal Flow as a Socially Aware Nash

A flow $f$ is optimal if and only if all flow travels along paths with minimum marginal cost

Marginal cost: $l_e(f_e) + f_e \cdot l_e'(f_e)$

A flow $f$ is at Nash equilibrium if and only if all flow travels along minimum latency paths

Latency: $l_e(f_e)$
Consequences for Linear Latency Fns

Observation: if \( l_e(f_e) = a_e f_e + b_e \)
(latency functions are linear) \( \Rightarrow \)
marginal cost of \( P \) w.r.t. \( f \) is:

\[
\sum_{e \in P} 2a_e f_e + b_e
\]

Corollary: if \( b_e = 0 \) for all \( e \), Nash and optimal flows coincide
Key Corollary

Corollary:

\[
\text{marginal costs of } f/2 = \text{latencies of } f
\]

\[
2a_e(f_e/2) + b_e = a_ef_e + b_e
\]

- \( f \) a Nash flow with rate \( r \) in a network with linear latency functions
  \( \Rightarrow f/2 \) is optimal with rate \( r/2 \)

\[
\text{common marginal cost} = \text{common latency of } f/2\text{'s paths} = \text{common latency of } f\text{'s paths}
\]

\[
:= L
\]
Goal: prove that cost of opt flow is at least $\frac{3}{4}$ times the cost of a Nash flow $f$

\[
\text{Cost of opt at rate } r = \text{Cost of opt at rate } \frac{r}{2} + \text{Cost of increasing rate from rate } \frac{r}{2} \text{ to rate } r
\]

opt is $\frac{f}{2}$

$C(f/2) \geq \frac{1}{4}C(f)$

At least $\left(\frac{r}{2}\right) \cdot L \geq \frac{1}{2}C(f)$
Extensions

• **Multicommodity networks**
  - many source-sink pairs

• **More general games**
  - proofs did not use structure provided by a network

• **Other classes of latency fns?**
  - proof for linear case is fragile
Sources of Inefficiency

**Corollary** of main Theorem:

- worst-case Nash/OPT ratio is realized on a two-link network!

Thus:

- one source of inefficiency:
  - confronted w/two routes, selfish users choose incorrectly

- **Corollary** $\Rightarrow$ that's all, folks!
  - network topology plays no role

![Diagram](image)

- Cost of *Nash* = 1
- Cost of *OPT* = $\frac{3}{4}$
No Dependence on Network Topology

**Theorem:** for (almost) any class of allowable latency fns, worst-case Nash/OPT ratio is realized on a two-link network.

**Corollary:** worst-case for bounded-degree polynomials is:

\[ x^k \]

\[ \begin{array}{c}
S & \xrightarrow{1} & t \\
1 & \xrightarrow{1-?} & 1 \\
0 & \xrightarrow{?} & ? \\
\end{array} \]
Part 2: Coping with Selfishness
Deleting Arcs to Improve a Nash Flow

Motivating Question: how can we "fix up" networks with a bad Nash flow?

![Diagram showing the network and the process of deleting arcs to improve the Nash flow.]
Designing Networks for Selfish Users

Formally:

- given network $G = (V,E,I)$
- find subnetwork minimizing latency experienced by all selfish users in a Nash flow

Def: The trivial algorithm is to build the entire network.
Designing Networks for Selfish Users is Hard

Our Results: the trivial algorithm is

• a $4/3$-approx alg with linear latency fns (follows from [RT00])

• an $n/2$-approx alg with general latency fns (new analysis needed)

and

• nothing better exists! (unless P=NP)

Corollary: in general, "bad edges" cannot be detected efficiently.
Motivation

**Goal:** prove that Nash flows are near-optimal
- want a laissez faire approach to regulating users

**Recall:** false in general!
Bad example: \((r=1, \ k \ \text{large})\)

Recall our previous solutions:
- relax notion of \(\text{OPT}\)
- restrict class of allowable latency fns
Taming Selfishness through a Manager

New Approach:
• not all traffic need be controlled by selfish users
  - “centrally controlled” vs. “selfishly controlled” traffic
  - behavior of selfish users depends on assignment of managed traffic

Goal:
• assign centrally controlled traffic to induce “good” selfish behavior
  - see also [Korilis, Lazar, Orda 97]
Stackelberg Strategies

- **Stackelberg strategy** = assignment of centrally controlled traffic
  \[\Rightarrow\] yields an induced equilibrium

- **Basic Questions:**
  - what’s the best strategy?
    - can we compute/characterize it?
  - how **inefficient** is the best induced equilibrium?
    - are we provably near-optimal?
A Constant-Factor Guarantee

Assumption: $G = \text{parallel links}$

Theorem: Can efficiently compute a strategy inducing an equilibrium with cost

$$= (1/\beta) \times \text{cost of opt flow}$$

($\beta = \text{fraction of managed traffic}$)

Fact: $(1/\beta) \times \text{OPT}$ is best possible.

Also: can get $[4/(3+\beta)] \times \text{OPT}$ for linear latency functions.
General Graphs

Open:

- for general latency fns, fixed $\beta$: a strategy inducing an equilibrium w/cost = $f(\beta) \times \text{opt}$
  - $1/\beta$ not achievable in general graphs (!)
  - maybe $2/\beta$? (or $O(n)$)

- for linear latency fns, a strategy w/cost < $4/3 \times \text{opt}$
  - e.g., is $8/7$ achievable for $\beta=\frac{1}{2}$?
Conclusions

**Moral:** you can be near-optimal in the presence of selfishness!

- Nash = $\frac{4}{3} \times \text{OPT}$ (linear latency fns)
  - similar results for other classes
  - multicommodity network no worse than a pair of parallel links
- Nash at rate $r = \text{OPT}$ at rate $2r$
- Stackelberg equilibrium = $\frac{1}{\beta} \times \text{OPT}$
  - with a good Stackelberg strategy
  - open: general networks?
Conclusions

Bad news: selfishness leads to increased complexity
  • intractability of network design

Big picture: other games?
  • we've studied only one simple model
    - Nash equilibria well-characterized
  • general themes:
    - are Nash equilibria near-optimal?
    - how to cope with selfishness?