Pricing Networks with Selfish Routing

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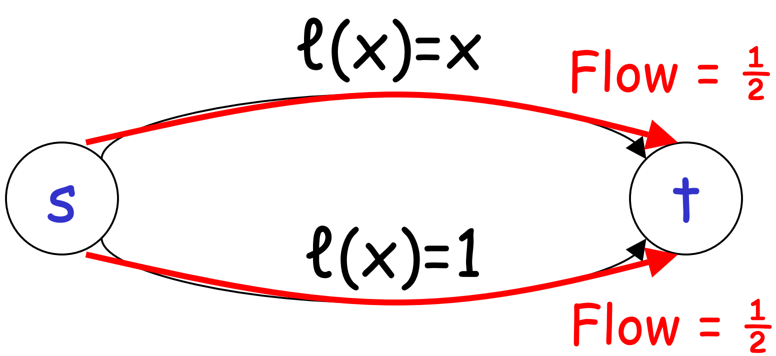
Survey of papers in STOC '03 and EC '03

Selfish Routing

- a directed graph $G = (V,E)$
- a source s and a destination t
- one unit of traffic from s to t
- for each edge e, a latency function $\ell_e(\cdot)$
  - assumed continuous, nondecreasing, convex

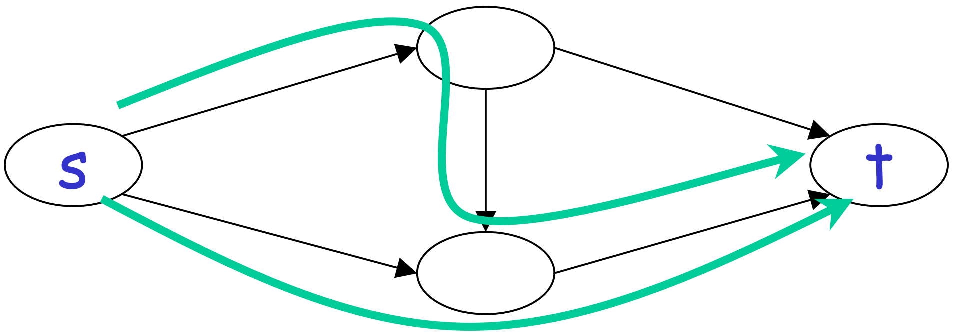
Example:



Routings of Traffic

Traffic and Flows:

- f_P = fraction of traffic routed on s-t path P
- flow vector **f** ⇔ routing of traffic



Selfish routing: what flows arise as the routes chosen by many noncooperative agents?

Nash Flows

Some assumptions:

- agents small relative to network
- want to minimize personal latency

Def: A flow is at Nash equilibrium (or is a Nash flow) if all flow is routed on min-latency paths [given current edge congestion]

- have existence, uniqueness [Wardrop, Beckmann et al 50s]

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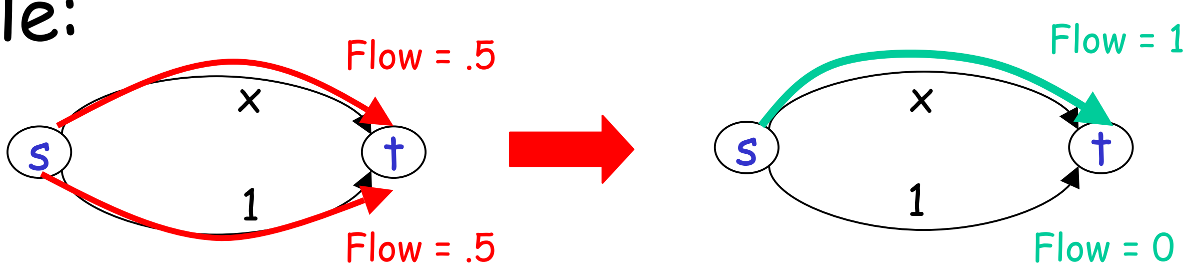
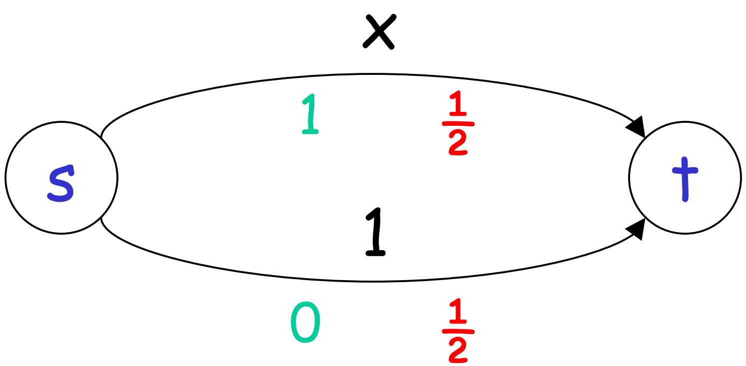


Inefficiency of Nash Flows

Our objective function: average latency

- ⇒ Nash flows need not be optimal
- observed informally by [Pigou 1920]



- Average latency of Nash flow = 1•1 + 0•1 = 1
- of optimal flow = $\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3}{4}$

Marginal Cost Taxes

Goal: do better with taxes (one per edge)

- not addressing implementation

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Thm: [folklore] marginal cost taxes w.r.t. the opt flow induce the opt flow as a Nash eq.

Why Homogeneous?

Problem: strong homogeneity assumption

- at odds with assumption of many users
- are taxes still powerful without this?

Our assumption: agent a has objective function time + $\beta(a)$ × money

- distribution function β assumed known
  - in aggregate sense

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Thm: can still induce opt flow as Nash eq, even with arbitrary heterogeneous users.

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Pf Idea: Brouwer's fixed-point thm.

- continuous fn on compact convex set has fixed pt
- want OPT-inducing taxes ⇔ fixed points

• continuous map:

- given tax vector not inducing OPT, push vector in helpful direction (else fixed pt)

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- Key Lemma: for sufficiently large bound, yes!
  - requires nontrivial proof (cf., Braess's Paradox)

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- in fact, set of all such taxes described by poly-sized list of linear inequalities
  - based on [Bergendorff et al 97]
  - can optimize secondary linear objective
- existence thm ⇒ there is a feasible point
  - otherwise set might be empty

When Taxes Cause Disutility

Problem #2 with MCT: min delay is holy grail; exorbitant taxes ignored

Question: are small taxes and min latency both possible?

- see also "frugal mechanisms" [Archer/Tardos]

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Thm: precise characterization of distribution functions β where both are always possible.

- strong condition, satisfied only with many misers

When Taxes Cause Disutility

Problem: what about for homogeneous traffic w/non-refundable taxes?

- e.g., when taxes are time delays

New Goal: minimize total disutility with non-refundable taxes (delay + taxes paid)

- call new objective fn the cost
- taxes can improve cost (Braess's Paradox)
- marginal cost taxes now not a good idea, e.g.:
- Thm: w/linear latency fns, MCT never help.

Taxes Are Powerful but Elusive

Thm: taxes can improve cost by a factor of $n/2$ ($n = |V|$), but no more.

- same for edge removal [Roughgarden FOCS '01]
- powerful, but can we compute them?

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Thm: no heuristic beats the trivial algorithm of assigning no taxes as all (unless P=NP).

- in the worst case
- complexity casts doubt on potential for taxes that minimize cost

My Favorite Open Question

Question: what remains true in multicommodity flow networks?

Note: Existence and algorithmic theorems for taxing heterogeneous traffic will hold if truncation trick still works.

- need “key lemma” that no bad fixed points exist