HORN EXTENSIONS OF A PARTIALLY DEFINED BOOLEAN FUNCTION

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Abstract. Given a partially defined Boolean function \((T,F)\) \((\text{pdBf})\), we investigate in this paper how to find a Horn extension \(f: \{0,1\}^n \rightarrow \{0,1\}\), which is consistent with \((T,F)\), where \(T \subseteq \{0,1\}^n\) denotes a set of true Boolean vectors (or positive examples) and \(F \subseteq \{0,1\}^n\) denotes a set of false Boolean vectors (or negative examples). Given a pdBf \((T,F)\), it is known that the existence of a Horn extension can be checked in polynomial time. As there are many Horn extensions, however, we consider those extensions \(f\) which have maximal and minimal sets \(T(f)\) of the true vectors of \(f\), respectively. For a pdBf \((T,F)\), there always exists the unique maximal (i.e., maximum) Horn extension, but there are in general many minimal Horn extensions. We first show that a polynomial time membership oracle can be constructed for the maximum extension, even if its disjunctive normal form (DNF) can be very long. Our main contribution is to show that checking if a given Horn DNF represents a minimal extension and generating a Horn DNF of a minimal Horn extension can both be done in polynomial time. We also can check in polynomial time if a pdBf \((T,F)\) has the unique minimal Horn extension. However, the problems of finding a Horn extension \(f\) with the smallest \(|T(f)|\) and of obtaining a Horn DNF, whose number of literals is smallest, are both NP-hard.

Key words. partially defined Boolean function, extension, Horn function, knowledge acquisition

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1. Introduction. Knowledge acquisition in the form of Boolean logic has been intensively studied (e.g., [2, 4, 6, 18, 20, 24]): Given a set of data, represented as a set \(T \subseteq \{0,1\}^n\) of binary “true \(n\)-vectors” (or “positive examples”) and a set \(F \subseteq \{0,1\}^n\) of “false \(n\)-vectors” (or “negative examples”), establish a (fully defined) Boolean function (i.e., extension) \(f: \{0,1\}^n \rightarrow \{0,1\}\) in a specified class \(C\), such that \(T \subseteq T(f)\) and \(F \subseteq F(f)\), where \(T(f)\) (resp., \(F(f)\)) denotes the set of true (resp., false) vectors of \(f\). A pair of sets \((T,F)\) is called a partially defined Boolean function (pdBf) throughout this paper.

For instance, a vector \(x\) may represent the symptoms used to diagnose a disease; e.g., \(x_1\) denotes whether temperature is high (=1) or not (=0), and \(x_2\) denotes whether blood pressure is high (=1) or not (=0), etc. Each vector \(x\) in \(T\) corresponds to a case of symptoms that caused the disease, while a vector in \(F\) describes a case with which the disease did not appear. Establishing an extension \(f\), which is consistent with the given data, amounts to finding a logical diagnostic explanation of the given data.

In this paper, we consider the case in which \(f\) is a Horn function [15]. The class of Horn functions is at the heart of knowledge-based systems [1, 12, 5] and motivates increasing research, e.g., minimum representations [13, 14], their learning
and identification [1, 8], and constructing Horn approximations [17, 25]. One of the main reasons for this attention is that the satisfiability problem (SAT) of a Horn conjunctive normal form (CNF) (H-SAT in short) can be solved in polynomial time [7], whereas the SAT of a general CNF is NP-complete [11]. As problem SAT of CNF is fundamental, many problems related to Horn functions can be solved efficiently.

In terms of sets $T(f)$ and $F(f)$, a Horn function has an elegant characterization: $f$ is Horn if and only if $F(f)$ is closed under intersection of vectors (i.e., $v, w \in F(f)$ implies $v \wedge w \in F(f)$, where $\wedge$ denotes the componentwise AND operation).

Because there are in general many Horn extensions $f$ for a given pdBf $(T, F)$, we shall mainly consider those extensions that are maximal and minimal in the sense of set $T(f)$, respectively. We note here that most of the papers written on the representation by Horn theory (e.g., [4, 8, 17, 18]) are based on model theory, in which finding a Horn representation $f$ of a given model $(T(g), F(g))$, where $g$ is a Boolean function and sets $T(g)$ and/or $F(g)$ of vectors are explicitly given, is a primary target. For example, the problem of finding the best Horn approximation of a model $(T(g), F(g))$, i.e., finding the Horn function with the minimum $|F(f)|$ under the constraint $F(f) \supseteq F(g)$, has received some attention [18], and it is known [18, 19] that obtaining an irredundant disjunctive normal form (DNF) of such an $f$ is at least as difficult as computing the DNF of the dual $h^d$ of a positive (i.e., monotone) Boolean function $h$. The latter problem is a well-known open problem [3, 9, 16], for which the recent result of Fredman and Khachiyan [10] shows that there is an $O(m^{o(\log m)})$ time algorithm, where $m$ is the total length of DNFs for both $h$ and $h^d$. We emphasize that our problem setting is different from model theory in that the input $(T, F)$ is only partially defined. However, the above problem of best Horn approximation is very close to the problem of finding a maximal Horn extension. We also note that, although the problem of finding a best approximation in terms of $T(g)$ is a bit artificial (since $T(g)$ is not closed under intersection), finding a minimal Horn extension of a pdBf $(T, F)$ is quite a natural problem in our framework.

It is known [4] that the existence of at least one Horn extension of a given pdBf $(T, F)$ can be checked in polynomial time. After preparing necessary notation and definitions in section 2 and introducing canonical Horn DNFs in section 3, we proceed to maximal and minimal Horn extensions. In section 4, by using an argument similar to the one used in model theory, we show that there exists the unique maximal Horn extension $f_{\text{max}}$ (i.e., maximum) and we provide a polynomial time membership oracle for $f_{\text{max}}$. In section 5, we investigate minimal Horn extensions. Contrary to the case of maximum Horn extension, there are in general many minimal Horn extensions. Our main contribution is to show that the minimality of $f_\varphi$, which denotes the function represented by a Horn DNF $\varphi$, can be checked in polynomial time. Based on this, a minimal Horn extension of a pdBf $(T, F)$ can be generated in polynomial time and the uniqueness of a minimal extension can also be checked in polynomial time.

To derive the above results, we first show that any minimal Horn extension can be represented by a canonical Horn DNF, although the converse is not true. The nontriviality of finding a canonical DNF representing a minimal Horn extension may be exemplified by the existence of a canonical DNF that satisfies local minimality but does not represent a minimal Horn extension. To overcome this, we reduce the nonminimality condition to the condition that some CNF $\Phi_v, v \in T$, is satisfiable, where $\Phi_v$ is a CNF derived from canonical DNF $\varphi, v \in T$, and $(T, F)$. Although this does not immediately give a polynomial time algorithm, since $\Phi_v$ is not Horn, we then derive a series of lemmas, with which (non-Horn) CNFs $\Phi_v$ can be eventually
transformed into Horn CNFs $\Phi^*_n$. Therefore, the minimality condition can be checked in polynomial time.

Finally, we show in sections 5 and 6 that the problems of computing a Horn extension $f$ with the minimum $|T(f)|$ and of finding the shortest Horn DNF (i.e., having the smallest number of literals) that represents a Horn extension are both NP-hard. It is still not known whether there exists a polynomial total time algorithm to generate all minimal Horn extensions of a given pdBF $(T, F)$.

2. Preliminaries. A Boolean function (or a function) is a mapping $f : \{0, 1\}^n \rightarrow \{0, 1\}$, where $x \in \{0, 1\}^n$ is called a Boolean vector (or a vector). If $f(x) = 1$ (resp., 0), then $x$ is called a true (resp., false) vector of $f$. The set of all true vectors (resp., false vectors) is denoted by $T(f)$ (resp., $F(f)$). Denote, for a vector $v \in \{0, 1\}^n$, $ON(v) = \{j \mid v_j = 1, j = 1, 2, \ldots, n\}$ and $OFF(v) = \{j \mid v_j = 0, j = 1, 2, \ldots, n\}$. For vectors $v, w \in \{0, 1\}^n$, we write $v \leq w$ (resp., $v \geq w$) if $v_i \leq w_i$ (resp., $v_i \geq w_i$) holds for all $i = 1, 2, \ldots, n$. Two special functions with $T(f) = \emptyset$ and $F(f) = \emptyset$ are, resp., denoted by $f = \top$ and $f = \bot$. For two functions $f$ and $g$ on the same set of variables, we write $f \leq g$ if $f(x) = 1$ implies $g(x) = 1$ for any $x \in \{0, 1\}^n$ and $f < g$ if $f \leq g$ and $f \neq g$.

Boolean variables $x_1, \ldots, x_n$ and their negations $\bar{x}_1, \ldots, \bar{x}_n$ are called literals, where we call literals $x_1, \ldots, x_n$ positive and literals $\bar{x}_1, \ldots, \bar{x}_n$ negative. A term $t$ is a conjunction of literals such that at most one of $x_i$ and $\bar{x}_i$ appears for each $i$. The constant 1 (viewed as the conjunction of an empty set of literals) is also considered a term. We say that a term $t$ subsumes a term $t'$ if $t \geq t'$, where terms $t$ and $t'$ are considered the functions they represent. For example, a term $x\bar{y}$ subsumes a term $x\bar{y}z$. A term $t$ is called an implicant of a function $f$ if $t \leq f$. An implicant $t$ of a function is called prime if there is no implicant $t' > t$.

A DNF $\varphi$ is a disjunction of terms. It is well known that a DNF $\varphi$ defines a function, which we denote by $f_\varphi$, and any function can be represented by a DNF (however, such a representation may not be unique). In this paper, we do not always distinguish a DNF $\varphi$ from the function $f_\varphi$ it represents. For example, a term $t$ is also considered as the function $f_t$. The number of literals in a DNF $\varphi$ is denoted by $|\varphi|$. In this paper, we shall deal exclusively with DNF expressions, although some of the literature on Horn functions is based on CNFs. By complementing the involved concepts, all the results in this paper can be translated into the results for CNFs.

A term is called positive if it contains only positive literals and is called Horn if it contains at most one negative literal. A DNF is called positive if it contains only positive terms and is called Horn if it contains only Horn terms. For example, a DNF $\varphi = 123 \lor 245 \lor 156$ is positive and $\psi = 157 \lor 24 \lor 267$ is Horn. (Here, for simplicity, a positive literal $x_i$ is denoted as $i$ and a negative literal $\bar{x}_i$ as $\bar{i}$.) It is easy to see that, by complementing Horn DNFs, we obtain Horn CNFs, where a CNF $\Phi = \bigwedge_i C_i$ is Horn if each clause $C_i$ contains at most one positive literal, e.g., $(1 \lor 2 \lor 3)(\bar{1} \lor 3)(\bar{1} \lor 3 \lor 4)$ is Horn, while $(1 \lor 2 \lor 3)(\bar{1} \lor 3)(\bar{1} \lor 3 \lor 4)$ is not Horn. A Boolean function is called positive (or monotone) if it can be represented by a positive DNF and Horn if it can be represented by a Horn DNF. It is known [14] that if $f$ is a Horn function, then all prime implicants of $f$ are Horn. It is important to know that the following two variants of SAT for a Horn CNF $\Phi$ can be solved in time linear in $|\Phi|$ [7, 22]:

**Problem H-SAT**

*Input:* A Horn CNF $\Phi$ of $n$ variables.

*Question:* Is there a vector $u \in \{0, 1\}^n$ satisfying $\Phi(u) = 1$?

**Problem UNIQUE-H-SAT**
Input: A Horn CNF \(\Phi\) of \(n\) variables.

Question: Is there a unique vector \(u \in \{0,1\}^n\) satisfying \(\Phi(u) = 1\)?

If the answer to these problems is “yes,” the vector \(u\) satisfying \(\Phi(u) = 1\) can also be output in linear time. If the answer to UNIQUE-H-SAT is “no,” two vectors \(u\) and \(v\) satisfying \(\Phi(u) = \Phi(v) = 1\) can also be output in linear time. Based on these, various problems associated with Horn functions can be solved in polynomial time. For example, given two Horn DNFs \(\varphi\) and \(\psi\), the conditions such as \(f_\varphi = f_\psi\) and \(f_\varphi < f_\psi\) can be checked in \(O(|\varphi||\psi|)\) time [14]. Also, for a term \(t\) (not necessarily Horn), condition \(t \leq f_\varphi\) can be checked in \(O(|\varphi|)\) time [14].

A pdBf is defined by a pair of sets \((T,F)\) satisfying \(T \cap F = \emptyset\), where \(T,F \subseteq \{0,1\}^n\). A function \(f\) is an extension (or theory) of the pdBf \((T,F)\) if \(T \subseteq T(f)\) and \(F \subseteq F(f)\), and it is a Horn extension if \(f\) is in addition Horn. We sometimes refer to a Horn DNF representing a Horn extension of a pdBf \((T,F)\) as a Horn DNF of \((T,F)\).

A Horn extension \(f\) of a pdBf \((T,F)\) is called minimal (resp., maximal) if there is no Horn extension \(f'\) satisfying \(f' < f\) (resp., \(f' > f\)), that is, set \(T(f)\) is minimal (resp., maximal). Furthermore, a Horn extension \(f\) of a pdBf \((T,F)\) is minimum (resp., maximum) if there is no Horn extension \(f'\) such that \(|T(f')| < |T(f)|\) (resp., \(|T(f')| > |T(f)|\)). Obviously, a minimum (resp., maximum) Horn extension is one of the minimal (resp., maximal) Horn extensions.

**Example 2.1.** Let \((T,F)\) be a pdBf defined by \(T = \{1110,0011,0101\}\) and \(F = \{0010,1100,0110\}\). Then, by generating all extensions of \((T,F)\), we can see that the unique maximum Horn extension of \((T,F)\) is represented by the DNF

\[
\varphi = 13 \lor 4 \lor 12
\]

while there are two minimal Horn extensions:

\[
\psi^{(1)} = 1234 \lor 134 \lor 124,
\]

\[
\psi^{(2)} = 1234 \lor 234 \lor 234.
\]

Furthermore, \(\psi^{(1)}\) represents a minimum Horn extension of \((T,F)\).

### 3. Canonical Horn DNF

In this section, we first review the following fundamental problem, which was originally discussed in [4]:

**Problem H-EXTENSION**

Input: A pdBf \((T,F)\).

Question: Is there a Horn extension \(f\) of \((T,F)\)?

We point out that the following well-known characterization of a Horn function provides a polynomial time algorithm to solve H-EXTENSION. Call the component-wise AND operation \(\land\) of vectors \(v\) and \(w\) the intersection of \(v\) and \(w\). For example, if \(v = (0101)\) and \(w = (1001)\), then \(v \land w = (0001)\). For a set \(X \subseteq \{0,1\}^n\), the set of vectors \(C(X)\) is called the intersection closure if it is a minimal set that contains \(X\) and is closed under intersection. Clearly, intersection closure is unique (i.e., “minimal” can be replaced by “minimum”).

**Proposition 3.1** (see [21, 8]). A function \(f\) is Horn if and only if \(F(f) = C(F(f))\) (i.e., \(F(f)\) is closed under intersection).

The next definition provides a means to generate Horn DNFs from \((T,F)\).

**Definition 3.1.** For a pdBf \((T,F)\) and a vector \(v \in T\), the set of terms \(R(v)\) is
defined by

\[ R(v) = \begin{cases} \{ \bigwedge_{j \in ON(v)} x_j \} & \text{if } OFF(v) = \emptyset, \\ \{ \bigwedge_{j \in ON(v)} x_j \} \setminus \{ l \in I(v) \} & \text{if } OFF(v) \neq \emptyset \text{ and } I(v) \neq \emptyset, \\ \emptyset & \text{if } OFF(v) \neq \emptyset \text{ and } I(v) = \emptyset, \end{cases} \]

where

\[ F_{\geq v} = \{ w \in F \mid w \geq v \}, \]
\[ I(v) = (\cap_{w \in F_{\geq v}} ON(w)) \cap OFF(v). \]

By convention, we define \( I(v) = OFF(v) \) if \( F_{\geq v} = \emptyset \). A DNF \( \varphi \) is called a canonical Horn DNF of \((T,F)\) if \( \varphi \) is given by

\[ \varphi = \bigvee_{v \in T} t_v, \text{ where } t_v \in R(v), \]

i.e., by selecting one term from each \( R(v), v \in T \). Note that the canonical Horn DNF is not defined if \( R(v) = \emptyset \) holds for some \( v \in T \).

For a \( v \in T \), there are many Horn terms \( t \) such that \( t(v) = 1 \). However, in order to satisfy \( t(w) = 0 \) for all \( w \in F \), we can restrict the negative literal \( \bar{x}_l \), which appears in \( t \). The above definition says that \( I(v) \) represents the set of such indices \( l \) and \( R(v) \) represents the particular subset of terms \( t \) such that \( t(v) = 1 \) and \( t(w) = 0 \) for all \( w \in F \). Construction of Horn DNFs in this manner can be found in the literature of learning theory [1], model theory [17], and Horn approximation [25]. Precisely speaking, however, the above canonical DNF is different from those used in the literature in that both \( T \) and \( F \) are explicitly taken into account.

Example 3.1. Let us define \( T, F \subseteq \{0,1\}^9 \) by

\[ T = \begin{cases} v^{(1)} = (111000100) \\ v^{(2)} = (111010100) \\ v^{(3)} = (110001010) \\ v^{(4)} = (001000100) \\ v^{(5)} = (100000100) \\ v^{(6)} = (011000001) \\ v^{(7)} = (110000001) \\ v^{(8)} = (111111000) \end{cases}, \]

\[ F = \begin{cases} w^{(1)} = (111001010) \\ w^{(2)} = (111010111) \\ w^{(3)} = (111001111) \\ w^{(4)} = (111000101) \end{cases}. \]

Then \( F_{\geq v^{(1)}} = \{ w^{(1)} \} \), \( F_{\geq v^{(2)}} = \{ w^{(2)} \} \), \( F_{\geq v^{(3)}} = \{ w^{(3)} \} \), \( F_{\geq v^{(4)}} = F_{\geq v^{(5)}} = \{ w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)} \} \), \( F_{\geq v^{(6)}} = F_{\geq v^{(7)}} = \{ w^{(2)}, w^{(4)} \} \), and \( F_{\geq v^{(8)}} = \emptyset \).

\[ I(v^{(1)}) = \{ 8 \}, \quad R(v^{(1)}) = \{ 123478 \}, \]
\[ I(v^{(2)}) = \{ 8, 9 \}, \quad R(v^{(2)}) = \{ 123578, 123579 \}, \]
\[ I(v^{(3)}) = \{ 7 \}, \quad R(v^{(3)}) = \{ 123678 \}, \]
\[ I(v^{(4)}) = \{ 1, 2 \}, \quad R(v^{(4)}) = \{ 137, 237 \}, \]
\[ I(v^{(5)}) = \{ 2, 3 \}, \quad R(v^{(5)}) = \{ 127, 137 \}, \]
\[ I(v^{(6)}) = \{ 1, 7 \}, \quad R(v^{(6)}) = \{ 1239, 2379 \}, \]
\[ I(v^{(7)}) = \{ 3, 7 \}, \quad R(v^{(7)}) = \{ 1239, 1279 \}, \]
\[ I(v^{(8)}) = \{ 7, 8, 9 \}, \quad R(v^{(8)}) = \{ 1234567, 1234568, 1234569 \}. \]

There are \( 1 \times 2 \times 1 \times 2 \times 2 \times 2 \times 2 \times 3 = 96 \) canonical Horn DNFs, among which we list the following two:

\[ \varphi^{(1)} = 123478 \vee 123579 \vee 123678 \vee 137 \vee 137 \vee 2379 \vee 1279 \vee 1234567, \]
\[ \varphi^{(2)} = 123478 \vee 123579 \vee 123678 \vee 137 \vee 137 \vee 1239 \vee 1239 \vee 1234569. \]
LEMMA 3.1 (see [4]). Any canonical Horn DNF \( \varphi \) of a given pdBf \((T,F)\) represents a Horn extension of \((T,F)\), and \((T,F)\) has no Horn extension if there is no canonical Horn DNF.

Proof. Let \( \varphi = \bigvee_{v \in T} t_v \) be a canonical Horn DNF of a pdBf \((T,F)\). It is clear that, for each \( v \in T \), we have \( t_v(v) = 1 \) and \( t_v(w) = 0 \) for all \( w \in F \). This implies that \( \varphi \) represents a Horn extension. Conversely, if there is no canonical Horn DNF, then \( R(v) = \emptyset \) holds for some \( v \in T \), i.e., \( \text{OFF}(v) \neq \emptyset \) and \( I(v) = \emptyset \) holds for some \( v \in T \). This means \( (\bigcap_{w \in F} \text{ON}(w)) \cap \text{OFF}(v) = \emptyset \); i.e., \( \bigwedge_{w \in F_{\not\in v}} w = v \). Therefore \( F(f) \) of no Horn extension \( f \) of \((T,F)\) is closed under intersection, and there is no Horn extension by Proposition 3.1. \( \square \)

Therefore, we have the following results.

**THEOREM 3.1** (see [4]). Problem \( H\text{-EXTENSION} \) can be solved in \( O(n||T||F|) \) time, and if a pdBf \((T,F)\) has a Horn extension, one of its canonical Horn DNFs can be obtained in \( O(n||T||F|) \) time.

Proof. The proof is immediate from the above discussion and the fact that a canonical Horn DNF of \((T,F)\) can be constructed in \( O(n||T||F|) \) time. \( \square \)

4. Maximum Horn extension. In this section, we first show the uniqueness of a maximal Horn extension.

**THEOREM 4.1.** If a given pdBf \((T,F)\) has a Horn extension, its maximal Horn extension is unique.

Proof. By Proposition 3.1, \( F(f) \) of any Horn extension \( f \) of \((T,F)\) is closed under intersection. Let us define \( f_{\max} \) by \( F(f_{\max}) = C(F) \), that is,

\[
(4.1) \quad f_{\max}(v) = \begin{cases} 
0 & \text{if } v \in C(F), \\
1 & \text{otherwise}.
\end{cases}
\]

Since \( C(F) \) is the unique minimal set that contains \( F \) and is closed under intersection, this \( f_{\max} \) is the unique maximal Horn extension (that is, \( T(f) \) is maximal) of \((T,F)\). \( \square \)

Unfortunately, it is known [17] that there is a pdBf \((T,F)\) for which the size of any DNF \( \varphi \) of \( f_{\max} \) is exponential in \( n, |T|, \) and \( |F| \). In other words, there may not be any compact DNF representation of \( f_{\max} \). However, we can do better if we do not stick to the DNF representation. Note that \( f_{\max} \) of (4.1) is defined by \( C(F) \), for which \( v \in C(F) \) holds if and only if

\[
\bigwedge_{w \in F_{\not\in v}} w = v.
\]

As this condition can be checked in polynomial time in \( n \) and \( |F| \) for a given \( v \), we can build an oracle that answers membership queries for \( f_{\max} \) in polynomial time.

A vector \( x \in X \subseteq \{0,1\}^n \) is called extreme [8] with respect to a set \( X \) if \( x \not\in C(X \setminus \{x\}) \). The set of all extremal vectors of \( X \) is called the characteristic set of \( X \) [17, 19] (or its base [8]) and is denoted by \( C^*(X) \). Note that every set \( X \subseteq \{0,1\}^n \) has the unique characteristic set \( C^*(X) \) and that \( C^*(X) \subseteq X \) is the minimum set satisfying \( C(C^*(X)) = C(X) \). It is known [8] that \( C^*(X) \) can be constructed from \( X \) in polynomial time in \( n \) and \( |X| \); therefore \( C^*(F) = C^*(C(F)) \) can be computed from \( F \) of \((T,F)\) in polynomial time. There are a number of papers on the relationship between \( C^*(F(f)) \) of a Horn function \( f \) and its Horn DNF expression \( \varphi \) [9, 17, 19]. For example, there is a polynomial total time algorithm (i.e., polynomial algorithm in the length of input and output) for computing from \( C^*(F(f)) \) all prime
implicants of Horn DNF $\varphi$ that represents $f$ if and only if there is a polynomial total time algorithm for dualizing a positive function $h$ (i.e., computing all prime implicants of $h^d$ from all prime implicants of $h$, where $h^d(x) = \bar{h}(\bar{x})$); if there is a polynomial total time algorithm for computing from $C^*(F(f))$ an irredundant Horn DNF $\varphi$ that represents $f$ (i.e., no term in $\varphi$ can be dropped), then there is a polynomial total time algorithm for dualizing a positive function. From the viewpoint of $f_{\text{max}}$ (whose $C^*(F(f_{\text{max}})) = C^*(F)$ can be computed in polynomial time), this shows that computing an irredundant Horn DNF $\varphi$ of $f_{\text{max}}$ is at least as hard as dualizing a positive function. It is not known yet [3, 9, 16] whether or not the problem of dualizing a positive function has a polynomial total time algorithm. However, the recent result by Fredman and Khachiyan [10] shows that dualizing a positive function can be done in $O(m^{\alpha (\log m)})$ time, where $m$ denotes the number of prime implicants of $f$ and $f^d$, and hence it is unlikely for the problem to be NP-hard.

5. Minimal Horn extensions. There are in general many minimal Horn extensions of a given pdBf $(T, F)$. However, these minimal Horn extensions can all have canonical Horn DNFs of Definition 3.1.

**Lemma 5.1.** A minimal Horn extension $f$ of a given pdBf $(T, F)$ can always be represented by a canonical Horn DNF.

**Proof.** Assume that there exists a minimal Horn extension $f$, which cannot be represented by a canonical DNF. Since $f$ is a Horn extension, for every $v \in T$, there is a Horn implicait $t_v = \bigwedge_{p \in P} x_i \bigwedge_{i \in N} \bar{x}_i$ of $f$ such that $t_v(v) = 1$ (i.e., $P \subseteq \text{OFF}(v)$ and $N \subseteq \text{ON}(v)$ and $|N| \leq 1$). Then by the definition of $R(v)$, there is a term $t'_v \in R(v)$ such that $t'_v \leq t_v$. Define a canonical Horn DNF $\varphi = \bigvee_{v \in T} t'_v$. This $\varphi$ satisfies $f_\varphi < f$ since $f_\varphi \leq f$ holds and $f$ cannot be represented by a canonical Horn DNF $\varphi$, contradicting the minimality of $f$. \hfill $\Box$

The converse, however, is not true (i.e., some canonical Horn DNFs do not represent minimal Horn extensions), as will be shown in Example 5.1 in the next subsection. Recall that a Horn DNF $\varphi$ representing a Horn extension of $(T, F)$ is called a Horn DNF of a pdBf $(T, F)$. Furthermore, we say that a Horn DNF $\varphi$ of $(T, F)$ is *minimal* if $f_\varphi$ is a minimal Horn extension of $(T, F)$. It is interesting to know whether the following problem can be solved in polynomial time, where we assume that a Horn DNF $\varphi$ (not necessarily canonical) is given as an input:

**Problem MINIMAL-H-EXTENSION**

**Input:** A pdBf $(T, F)$ and a Horn DNF $\varphi$.

**Question:** Is $\varphi$ a minimal Horn DNF of $(T, F)$?

In passing, we note an interesting implication of Lemma 5.1: all minimal Horn extensions of $(T, F)$ have “short” DNFs in the sense that all canonical Horn DNFs have only $|T|$ terms, respectively. This contrasts with the fact that the DNFs of some maximum Horn extensions have exponentially many terms, as noted after Theorem 4.1.

5.1. Checking the minimality of a Horn DNF. We show via a series of lemmas in this subsection that MINIMAL-H-EXTENSION can be solved in polynomial time. In the following, we assume without loss of generality that $(1, 1, \ldots, 1) \notin T$ holds, because it can be shown that, for a pdBf $(T, F)$ with $(1, 1, \ldots, 1) \in T$, $\varphi \lor (\bigwedge_{j=1}^{n} x_j)$ is a minimal Horn DNF of $(T, F)$ if and only if $\varphi$ is a minimal Horn DNF of $(T \setminus \{(1, 1, \ldots, 1)\}, F)$. For a pdBf $(T, F)$, a vector $v \in T$, and a Horn DNF
Lemma 5.1, if a Horn DNF $\varphi$ is generated from 1¯34¯5 and ¯12¯346. It is known [23] that all prime implicants of $\varphi$ can be generated by the consensus procedure starting from the terms of $\varphi$.

Note that Lemma 5.1 implies $I(v; \varphi) \neq \emptyset$ and $R(v; \varphi) \neq \emptyset$ for all $v \in T$. By Lemma 5.1, if a Horn DNF $\varphi$ of $(T, F)$ is not minimal, then there exists a canonical Horn DNF $\psi$ of $(T, F)$ such that $f_\psi < f_\varphi$, where $\psi$ can be written as

$$\psi = \bigvee_{v \in T} t_v; \quad t_v \in R(v; \varphi).$$

It is known that the candidate set of terms $R(v; \varphi)$ can be computed in polynomial time. More precisely, given a canonical Horn DNF $\varphi$ and a vector $v \in T$, set $I(v; \varphi)$ can be constructed in time linear in $|\varphi|$ by using the following forward chaining procedure [13].

**Algorithm F-CHAINING**

Input: A Horn DNF $\varphi$ and a vector $v \in T$.

Output: Set $I(v; \varphi)$.

**Step 1:** $S := ON(v)$ and $I(v; \varphi) := \emptyset$.

**Step 2:** If there exists a term $t = (\bigwedge_{i \in S} x_i)\bar{x}_l$ in $\varphi$ such that $S' \subseteq S$ and $l \notin S$,

let $S := S \cup \{l\}$. Repeat Step 2 until no term in $\varphi$ satisfies the condition.

**Step 3:** $I(v; \varphi) := S \setminus ON(v)$.

The essential part of this algorithm comes from the consensus procedure [23]. Given a DNF $\varphi$, the consensus procedure generates a new implicant $\bigwedge_{j \in P_k} x_j \bigwedge_{j \in N_k} \bar{x}_j$ from two implicants $\bigwedge_{j \in P_k} x_j \bigwedge_{j \in N_k} \bar{x}_j$ and $\bigwedge_{j \in P_k} x_j \bigwedge_{j \in N_k} \bar{x}_j$ such that $i \notin P_k \cup N_k$ for $k = 1, 2$, and $P_1 \cap N_2 = N_1 \cap P_2 = \emptyset$. For example, 23456 is generated from 1345 and 12346. It is known [23] that all prime implicants of $f_\varphi$ eventually can be generated by the consensus procedure starting from the terms of $\varphi$. Since every prime implicant of a Horn function is Horn, Algorithm F-CHAINING works correctly.

**Example 5.1.** Consider the pdBf $(T, F)$ given in Example 3.1 and choose two canonical Horn DNFs $\varphi(1)$ and $\varphi(2)$ of (3.2). Then

$$I(v(1); \varphi(1)) = \{8\}, \quad I(v(2); \varphi(1)) = \{9\},$$

$$I(v(3); \varphi(1)) = \{7\}, \quad I(v(4); \varphi(1)) = \{1\},$$

$$I(v(5); \varphi(1)) = \{3\}, \quad I(v(6); \varphi(1)) = \{1, 7\},$$

$$I(v(7); \varphi(1)) = \{3, 7\}, \quad I(v(8); \varphi(1)) = \{7, 8, 9\},$$

and

$$R(v(1); \varphi(1)) = \{123478\}, \quad R(v(2); \varphi(1)) = \{123579\},$$

$$R(v(3); \varphi(1)) = \{123678\}, \quad R(v(4); \varphi(1)) = \{137\},$$

$$R(v(5); \varphi(1)) = \{137\}, \quad R(v(6); \varphi(1)) = \{1239, 2379\},$$

$$R(v(7); \varphi(1)) = \{1239, 1279\}, \quad R(v(8); \varphi(1)) = \{1234567, 1234568, 1234569\},$$

$$R(v(1); \varphi(2)) = \{8\}, \quad R(v(2); \varphi(2)) = \{9\},$$

$$R(v(3); \varphi(2)) = \{7\}, \quad R(v(4); \varphi(2)) = \{1\},$$

$$R(v(5); \varphi(2)) = \{3\}, \quad R(v(6); \varphi(2)) = \{1\},$$

$$R(v(7); \varphi(2)) = \{3\}, \quad R(v(8); \varphi(2)) = \{9\}.$$
The two functions \( f_{\varphi(1)} \) and \( f_{\varphi(2)} \) satisfy \( f_{\varphi(1)} \geq f_{\varphi(2)} \), since both are canonical and \( R(v(l); \varphi(1)) \geq R(v(l); \varphi(2)) \) holds for all \( l \). Furthermore, \( 2379 \leq f_{\varphi(1)} \) and \( 2379 \leq f_{\varphi(2)} \) imply \( f_{\varphi(1)} > f_{\varphi(2)} \). Therefore, \( \varphi(1) \) is not minimal. However, this \( \varphi(1) \) satisfies local minimality in the sense that, after replacing one of its terms \( t_v \) by \( t'_v \in R(v(l); \varphi(1)) \setminus \{ t_v \} \) for any \( l \) with \( |R(v(l); \varphi(1))| > 1 \), the resulting DNF also represents \( f_{\varphi(1)} \). For example, the following two DNFs also represent the same \( \varphi \):

\[
\varphi^{(3)} = 123478 \vee 123579 \vee 123678 \vee 137 \vee 1279 \vee 1279 \vee 129 \vee 1279 \vee 12345678,
\]

\[
\varphi^{(4)} = 123478 \vee 123579 \vee 123678 \vee 137 \vee 1279 \vee 1239 \vee 1279 \vee 12345677.
\]

This result shows that local minimality of \( \varphi \) (in the above sense) does not always imply its minimality. Therefore, some other proof is necessary to ensure the minimality. Of course, if we replace more than one term in \( \varphi^{(1)} \), the resulting DNF may represent a different function; \( \varphi^{(2)} \) is such an example.

On the other hand, \( \varphi^{(2)} \) is minimal since \( |R(v(l); \varphi^{(2)})| = 1 \) for all \( l = 1, 2, \ldots, 8 \).

However, this is not always the case, since there can be a minimal \( \varphi \) with \( |R(v(l); \varphi)| > 1 \) for some \( l \). For example, consider a pdBf \( (T, F) \) defined by \( T = \{ v^{(1)} = (1100), v^{(2)} = (1010), v^{(3)} = (0110) \} \) and \( F = \emptyset \), and a canonical DNF \( \varphi = 123 \vee 134 \vee 234 \). Then \( I(v^{(1)}; \varphi) = \{ 3, 4 \}, I(v^{(2)}; \varphi) = \{ 4 \}, \) and \( I(v^{(3)}; \varphi) = \{ 3 \} \); that is, \( R(v^{(1)}; \varphi) = \{ 123, 124 \}, R(v^{(2)}; \varphi) = \{ 134 \}, \) and \( R(v^{(3)}; \varphi) = \{ 234 \} \). However, since it is easy to see that \( \varphi' = 124 \vee 134 \vee 234 \) satisfies \( f_{\varphi'} = f_{\varphi} \), there is no canonical DNF \( \psi \) such that \( \psi < \varphi \); hence \( \varphi \) is minimal.

Example 5.1 may suggest that Problem MINIMAL-H-EXTENSION is not trivial.

Let us now examine the condition when \( \varphi \) is not minimal.

Let \( f \) be a Horn extension of \( (T, F) \). Then \( f \) is not minimal if and only if there is a nonempty subset of \( T(f) \setminus T \), whose removal from \( T(f) \) results in a new Horn extension. This means that there exists a vector \( u \in T(f) \) such that \( C(F(f) \cup \{ u \}) \cap T = \emptyset \) holds, where \( C(X) \) denotes the intersection closure of \( X \). In other words, \( u \wedge \bigwedge_{w \in S} w \) is different from any vector in \( T \) for all \( S \subseteq F(f) \). Since \( u \wedge \bigwedge_{w \in S} w = a \) holds for some \( a \in T \) and \( S \subseteq F(f) \) if and only if it holds for \( S = F(f)_{\geq a} \), where \( F(f)_{\geq a} = \{ w \in F(f) \mid w \geq a \} \), this argument leads to the following lemma.

**Lemma 5.2.** Let \( f \) be a Horn extension of a pdBf \( (T, F) \). Then \( f \) is not minimal if and only if there exists a vector \( u \in T(f) \) such that

\[
(5.1) \quad u \wedge \bigwedge_{w \in F(f)_{\geq a}} w \neq a \quad \text{for all } a \in T.
\]

Assume that \( f \) can be represented by a canonical Horn DNF \( \varphi \) of a pdBf \( (T, F) \) (i.e., \( f = f_{\varphi} \)). Then \( y = \bigwedge_{w \in F(f)_{\geq a}} w \) for an \( a \in T \) is given by \( ON(y) = ON(a) \cup I(a; \varphi) \), because, by the definition of \( I(a; \varphi) \), all vectors \( w \) such that \( w \geq a \) and \( w_l = 0 \) for some \( l \in I(a; \varphi) \) satisfy \( \varphi(w) = 1 \) (i.e., \( w \notin F(f) \)), and, for every \( l \in OFF(a) \setminus I(a; \varphi) \), there is a vector \( w \in F(f)_{\geq a} \) such that \( w_l = 0 \) (since otherwise \( l \) must be included in \( I(a; \varphi) \)).

In other words, \( u \wedge \bigwedge_{w \in F(f)_{\geq a}} w = a \) holds for an \( a \in T \) if and only if \( u \) satisfies \( ON(u) \supseteq ON(a) \) and \( OFF(u) \supseteq I(a; \varphi) \). Thus, the condition (5.1) in Lemma 5.2 can be rewritten as follows.

**Lemma 5.3.** Let \( \varphi = \bigvee_{v \in T} t_v \) be a canonical Horn DNF of a pdBf \( (T, F) \). Then \( \varphi \) is not minimal if and only if at least one of the following CNFs is satisfiable:

\[
(5.2) \quad \Phi_v = t_v \wedge \bigwedge_{a \in T} C_a, \quad v \in T,
\]
where

\[
C_a = \left( \bigvee_{j \in ON(a)} \bar{x}_j \lor \bigvee_{j \in I(a; \varphi)} x_j \right).
\]

That is, there is a vector \( u \in \{0, 1\}^n \) such that \( \Phi_v(u) = 1 \) for some \( v \in T \).

**Proof.** By the above discussion, \( \varphi \) is not minimal if and only if the formula

\[
\varphi(x) \land \bigwedge_{a \in T} \left( \bigvee_{j \in ON(a)} \bar{x}_j \lor \bigvee_{j \in I(a; \varphi)} x_j \right)
\]

is satisfiable, which is equivalent to (5.2). \( \square \)

**Example 5.2.** Consider the pdBf \( (T, F) \) of Example 3.1. Recall that we have the following canonical DNF:

\[
\varphi = \varphi^{(1)} = 123478 \lor 123579 \lor 123678 \lor 137 \lor 137 \lor 2379 \lor 1279 \lor 1234567.
\]

Using \( I_j^{(1)}, \varphi^{(1)} \), \( l = 1, 2, \ldots, 8 \), listed in Example 5.1, (5.2) can be written as

\[
\Phi_v^{(a)} = 123478(1 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 4 \lor 5 \lor 8 \lor 9)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

\[
\Phi_v^{(a)} = 123478(1 \lor 2 \lor 3 \lor 4 \lor 7 \lor 8)(1 \lor 2 \lor 3 \lor 5 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 6 \lor 7 \lor 8)(1 \lor 3 \lor 7)
\]

\[
(1 \lor 3 \lor 7)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9)(1 \lor 2 \lor 3 \lor 7 \lor 9),
\]

Now take a vector \( u = (111000001) \). This \( u \) satisfies \( \Phi_v^{(a)}(u) = 1 \), showing that \( f_v \) is not a minimal Horn extension of \( (T, F) \).

Note that CNFs \( \Phi_v \) of (5.2) are not Horn, in general, and therefore their SATs in Lemma 5.3 may not be easy. However, we prove in the rest of this subsection that this can be done in polynomial time.

Let \( \varphi = \bigvee_{v \in T} t_v \) be a canonical Horn DNF of \( (T, F) \). Given a \( v \in T \), define

\[
\hat{l}(v; \varphi) = I(v; \varphi) \setminus \{l_v\},
\]

where \( \bar{x}_v \) is the negative literal in \( t_v \). Now assume that \( \Phi_v \) defined by (5.2) is satisfiable, i.e., there is a vector \( u \in \{0, 1\}^n \) such that \( \Phi_v(u) = 1 \). This means \( t_v(u) = 1 \), and hence

\[
ON(u) \supseteq ON(v) \text{ and } l_v \in OFF(u).
\]

Therefore, we can fix \( u_j = 1 \) for all \( j \in ON(v) \) and \( u_{l_v} = 0 \). We also have

\[
ON(u) \cap \hat{l}(v; \varphi) \neq \emptyset
\]
in order to satisfy the clause \((\bigvee_{j \in ON(v)} \bar{x}_j \vee \bigvee_{j \in I(v; \varphi)} x_j)\) associated with \(a = v\). Furthermore, we shall show below that we can fix
\begin{equation}
(5.7) \quad u_j = 0 \text{ for all } j \in OFF(v) \setminus I(v; \varphi)
\end{equation}
without loss of generality. As a result of these observations, we denote by
\begin{equation}
(5.8) \quad \Phi'_a = \bigwedge_{a \in T} C'_a
\end{equation}
the CNF obtained from \(\Phi_v\) by fixing variables \(x_j\) to 1 for \(j \in ON(v)\) and 0 for \(j \in OFF(v) \setminus \hat{I}(v; \varphi)\), in which \(C'_a\) denotes the clause obtained from \(C_a\) in the same way. Then \(\Phi_v\) is satisfiable if and only if \(\Phi'_v\) is satisfiable. Note that only those \(x_j\) satisfying \(j \in \hat{I}(v; \varphi)\) remain as variables in \(\Phi'_v\). In Example 5.2, it can be seen that \(\Phi'_{v(i)} = \bot\) for \(i = 1, 2, 3, 4, 5\), \(\Phi'_{v(6)} = 1\), \(\Phi'_{v(7)} = 3\), and \(\Phi'_{v(8)} = 8(8 \lor 9)\). This may indicate that \(\Phi'_a\) is much simpler than \(\Phi_v\) to consider.

Now we prove the above claim (5.7) after showing the next lemma.

**Lemma 5.4.** Let \(\varphi = \bigvee_{v \in T} \iota_v\) be a canonical Horn DNF of a pdBF \((T, F)\). Let \(\Phi_v = \iota_v \land \bigwedge_{a \in T} C_a\) and \(\Phi'_v = \bigwedge_{a \in T} C'_a\) be defined as above for a \(v \in T\). Then \(C'_a \neq \top\) holds for an \(a \in T\) if and only if the following two conditions hold:
1. \(ON(a) \subseteq ON(v) \cup \hat{I}(v; \varphi)\),
2. \(I(a; \varphi) \subseteq I(v; \varphi)\).

**Proof.** It is easy to see that if a satisfies (i) and (ii), then \(C'_a \neq \top\) holds because \(C'_a\) is obtained from \(C_a = (\bigvee_{j \in ON(a)} \bar{x}_j \vee \bigvee_{j \in I(a; \varphi)} x_j)\) by fixing \(x_j = 1\) for \(j \in ON(v)\) and 0 for \(j \in OFF(v) \setminus \hat{I}(v; \varphi)\).

On the other hand, let us assume that \(C'_a \neq \top\) holds. If there exists an \(l \in ON(a) \setminus (ON(v) \cup \hat{I}(v; \varphi))\) \((= ON(a) \cap (OFF(v) \setminus \hat{I}(v; \varphi)))\), then \(C'_a = \top\) holds, because the \(x_l\) is fixed to 0, which is a contradiction to the assumption. This proves property (i).

Next, to prove (ii), assume that there exists an index \(l \in I(a; \varphi) \setminus I(v; \varphi)\). The following two cases are possible:

(a) \(l \in ON(v) \cap I(a; \varphi)\). Then, \(x_l\) is fixed to 1 and \(C'_a = \top\) holds, which is a contradiction.

(b) \(l \in OFF(v) \cap I(a; \varphi)\). Clearly, \(l \in (OFF(v) \setminus I(v; \varphi)) \cap I(a; \varphi)\). Then \(l \in I(a; \varphi)\) implies \((\bigwedge_{j \in ON(a)} x_j) \bar{x}_l \leq f_\varphi\), and therefore, by property (i),
\begin{equation}
(5.9) \quad \left( \bigwedge_{j \in ON(v) \cup I(v; \varphi)} x_j \right) \bar{x}_l \leq f_\varphi.
\end{equation}

Now, \(l \in OFF(v) \setminus I(v; \varphi)\) implies that \((\bigwedge_{j \in ON(v)} x_j) \bar{x}_l \not\leq f_\varphi\). However, since \((\bigwedge_{j \in ON(v)} x_j) \bar{x}_l \leq f_\varphi\) for all \(h \in I(v; \varphi)\) and
\[
T \left( \left( \bigwedge_{j \in ON(v) \cup I(v; \varphi)} x_j \right) \bar{x}_l \right) = T \left( \left( \bigwedge_{j \in ON(v)} x_j \right) \bar{x}_l \right) \setminus T \left( \left( \bigwedge_{h \in I(v; \varphi)} x_h \right) \bar{x}_l \right),
\]
we have \((\bigwedge_{j \in ON(v) \cup I(v; \varphi)} x_j) \bar{x}_l \not\leq f_\varphi\), which is a contradiction to (5.9).  \(\square\)
Now, we prove our claim.

**Lemma 5.5.** Let \( \varphi = \bigvee_{v \in T} t_v \) be a canonical Horn DNF of a pdBf \((T, F)\). If a vector \( u \) satisfies \( \Phi_v(u) = 1 \) for a vector \( v \in T \), then the vector \( u' \) also satisfies \( \Phi_v(u') = 1 \), where \( u' \) is defined by

\[(5.10) \quad u'_j = \begin{cases} 
  u_j & \text{if } j \in ON(v) \cup I(v; \varphi), \\
  0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Let \( \Phi'_v = \bigwedge_{a \in T} C'_a \) and consider an \( a \in T \) such that \( C'_a \neq T \). By \( \Phi_v(u) = 1 \), \( C_a(u) = \left( \bigvee_{j \in ON(a)} u_j \right) \cup \bigvee_{j \in I(a; \varphi)} u_j = 1 \) holds. However, considering the condition \( ON(a) \cup I(a; \varphi) \subseteq ON(v) \cup I(v; \varphi) \) (which follows from Lemma 5.4), we have \( C_a(u') = C_a(u) = 1 \). Furthermore, \( C_b(u') = 1 \) holds for all other clauses \( C_b \) with \( C'_b = T \), and also \( t_v(u') = 1 \) holds. These prove \( \Phi_v(u') = 1 \). \( \Box \)

Now, we summarize the above result as the following lemma.

**Lemma 5.6.** Let \( \varphi = \bigvee_{v \in T} t_v \) be a canonical Horn DNF of a pdBf \((T, F)\). Then \( \varphi \) is not minimal if and only if at least one of the CNFs \( \Phi'_v, v \in T \), is satisfiable, where \( \Phi'_v \) is defined by (5.8).

In order to find a vector \( u \) such that \( \Phi'_v(u) = 1 \), we can remove from \( \Phi'_v \) all the clauses \( C'_a = T \). Furthermore, by (5.6), if a vector \( u \) satisfies \( \Phi'_v(u) = 1 \), then all other clauses \( C'_a \) such that \( I(a; \varphi) = I(v; \varphi) \) satisfy \( C'_a(u) = 1 \). Therefore, we can also remove from \( \Phi'_v \) all the clauses \( C'_a \) satisfying \( a \neq v \) and \( I(a; \varphi) = I(v; \varphi) \). In the following, we write the resulting CNF also as \( \Phi'_v \). In other words, denoting by \( T_v \) the set of vectors \( a \in T \) such that \( C'_a \neq T \) and \( I(a; \varphi) \subseteq I(v; \varphi) \), we have \( \Phi'_v(u) = 1 \) if and only if \( u \notin T_v \) holds by definition, and that a vector \( u \) satisfies \( C'_a(u) = 1 \) if and only if \( ON(u) \cup I(v; \varphi) \neq \emptyset \) holds (since \( C'_v = \bigvee_{j \in I(v; \varphi)} x_j \)). Since \( C'_v \) is a special clause in \( \Phi'_v \) (see the subsequent discussion), we check the conditions \( C'_v(u) = 1 \) and \( \Phi'_v(u) = 1 \) separately, where \( \Phi'_v = \bigwedge_{a \in T_v} C'_a \). We emphasize here that these CNFs \( \Phi'_v \) may still be non-Horn. However, the following lemma shows that they can be transformed into Horn CNFs

\[(5.11) \quad \Phi'_v = C'_v \land \bigwedge_{a \in T_v} C'_a.
\]

Note that \( v \notin T_v \) holds by definition, and that a vector \( u \) satisfies \( C'_a(u) = 1 \) if and only if \( ON(u) \cup I(v; \varphi) \neq \emptyset \) holds (since \( C'_v = \bigvee_{j \in I(v; \varphi)} x_j \)). Since \( C'_v \) is a special clause in \( \Phi'_v \) (see the subsequent discussion), we check the conditions \( C'_v(u) = 1 \) and \( \Phi'_v(u) = 1 \) separately, where \( \Phi'_v = \bigwedge_{a \in T_v} C'_a \). We emphasize here that these CNFs \( \Phi'_v \) may still be non-Horn. However, the following lemma shows that they can be transformed into Horn CNFs

\[(5.12) \quad \Phi'_v = \bigwedge_{a \in T_v} C'_a,
\]

where \( C'_a \) denotes the clause obtained from \( C'_a \) by removing all literals \( x_j, j \in I(a; \varphi) \). For example, if \( C'_a = (1 \lor 2 \lor 3 \lor 4 \lor 5) \) and \( I(a; \varphi) = \{4, 5, 6\} \), then \( C'^*_a = (1 \lor 2 \lor 3) \) holds; in this case \( \bar{x}_3 \) is the negative literal in \( t_a \). We can easily see that \( \Phi'^*_v \) is in fact Horn, because each clause \( C'_a \) has at most one positive literal \( x_{a_{ij}} \), which appears negated in \( t_a \).

**Lemma 5.7.** Let \( \varphi = \bigvee_{v \in T} t_v \) be a canonical Horn DNF of a pdBf \((T, F)\). Then \( \varphi \) is not minimal if and only if at least one of the Horn CNFs \( \Phi'^*_v, v \in T \), has a vector \( u \in \{0, 1\}^n \) such that \( \Phi'^*_v(u) = 1 \) and \( ON(u) \cap I(v; \varphi) = \emptyset \).

**Proof.** Let us first assume that some \( \Phi'^*_v \) has a vector \( u \) such that \( \Phi'^*_v(u) = 1 \) and \( ON(u) \cap I(v; \varphi) = \emptyset \). Since \( \Phi'^*_v(u) = 1 \) and \( ON(u) \cap I(v; \varphi) = \emptyset \), resp., imply \( C'_a(u) = 1 \) for all \( a \in T_v \), and \( C'_v(u) = 1 \) (note that \( C'_v = \bigvee_{j \in I(v; \varphi)} x_j \)), we have \( \Phi'_v(u) = 1 \). Thus Lemma 5.6 shows the if-part.
To prove the only-if part, let us assume by Lemma 5.6 that \( \Phi'_v \) has a vector \( u \) such that \( \Phi'_v(u) = 1 \) and has the minimum \( |I(v; \varphi)| \); i.e., no \( \Phi'_v \) satisfies \( |I(w; \varphi)| < |I(v; \varphi)| \). By (5.6), this \( u \) must satisfy \( ON(u) \cap \hat{I}(v; \varphi) \neq \emptyset \). To show \( \Phi'_v(u) = 1 \), let us assume the contrary, i.e., \( C'_b(u) = 0 \) holds for some \( b \in T_v \). Without loss of generality, we assume that the condition

\[
(5.13) \quad u_j = 1 \text{ for all } j \in ON(v) \text{ and } 0 \text{ for all } j \in OFF(v) \setminus \hat{I}(v; \varphi)
\]

holds, since \( \Phi'_v \) consists of only those variables \( x_j \) satisfying \( j \in \hat{I}(v; \varphi) \). Then \( C'_b(u) = 0 \) implies \( \bar{t}_b(u) = (\bigvee_{j \in ON(b)} \bar{a}_j \lor u_j) = 0 \), where \( \bar{x}_b \) is the negative literal in \( t_b \). This is because \( C'_b \) is obtained from the clause \( (\bigvee_{j \in ON(b)} x_j \lor x_0) \) by fixing \( x_j = 1 \) for \( j \in ON(v) \) and 0 for \( j \in OFF(v) \setminus \hat{I}(v; \varphi) \) (which \( u \) also satisfies by (5.13)). Thus we have \( t_b(u) = 1 \). Since \( \Phi_v = t_v \land \bigwedge_{a \in T} C_a \) and \( \Phi_b = t_b \land \bigwedge_{a \in T} C_a \), \( t_b(u) = 1 \) and \( \Phi_b(u) = 1 \) imply \( \Phi_b(u) = 1 \). However, by Lemmas 5.3 and 5.6, this means that \( \Phi'_b \) is satisfiable, which is a contradiction to our assumption that \( |I(v; \varphi)| \) is the minimum (since \( I(b; \varphi) \subset I(v; \varphi) \) holds by (ii) of Lemma 5.4 and the discussion following Lemma 5.6).

Based on Lemma 5.7, we can propose an algorithm to solve problem MINIMAL-H-EXTENSION.

**Algorithm CHECK-MINIMAL**

**Input:** A pdBF \((T, F)\) and a Horn DNF \( \varphi \).

**Question:** Is \( \varphi \) a minimal Horn DNF of \((T, F)\)?

**Step 1:** Check if \( f_\varphi \) is a Horn extension of \((T, F)\). If not, output “no” and halt.

**Step 2:** Construct a canonical Horn DNF \( \psi = \bigvee_{v \in T} t_v \) such that \( f_\psi \leq f_\varphi \). If \( f_\psi = f_\varphi \), then output “no” and halt; otherwise (i.e., \( f_\psi < f_\varphi \)), rewrite \( \psi \) as \( \varphi \).

**Step 3:** For each \( v \in T \), check if \( \Phi'_v \) has a vector \( u \) such that \( ON(u) \cap \hat{I}(v; \varphi) \neq \emptyset \) and \( \Phi'_v(u) = 1 \). Output “no” if some \( \Phi'_v \) has such a vector; otherwise, “yes.” Halt.

**Theorem 5.1.** Given a pdBF \((T, F)\) and a Horn DNF \( \varphi \), Problem MINIMAL-H-EXTENSION can be solved in \( O(|F| |\varphi| + n|T| |\varphi| + n|T|^2) \) time by Algorithm CHECK-MINIMAL, where \( T, F \subseteq \{0, 1\}^n \) and \( |\varphi| \) denotes the number of literals in \( \varphi \).

**Proof.** The correctness of Algorithm CHECK-MINIMAL follows from Lemma 5.7. We therefore consider its time complexity. Step 1 can be executed in \( O(|F| |\varphi|) \) time, since we can check if \( \varphi(v) = 1 \) or 0 for each \( v \in T \cup F \) in \( O(|\varphi|) \) time. In Step 2, we first compute \( I(v; \varphi) \) for all \( v \in T \), which can be done in \( O(|F| |\varphi|) \) time by applying Algorithm F-CHAINING in subsection 5.1 to all \( v \in T \). Choose an arbitrary term \( t_v \in R(v; \varphi) \) for each \( v \in T \), and then construct a canonical Horn DNF \( \psi = \bigvee_{v \in T} t_v \), which satisfies \( f_\psi \leq f_\varphi \) and \( |\psi| \leq n|T| \). This can be done in \( O(n|T|) \) time. Checking if two Horn DNFs \( \psi \) and \( \varphi \) satisfy \( f_\psi < f_\varphi \) can then be done in \( O(|\psi| |\varphi|) = O(n|T| |\varphi|) \) time [14]. Totally, Step 2 requires \( O(n|T| |\varphi|) \) time. In Step 3, for each \( v \in T \), we can construct \( \Phi'_v \) in \( O(n|T|) \) time, because \( I(v; \varphi) \) was already obtained in Step 2.

Let us now consider how to check if there is a vector \( u \) such that \( ON(u) \cap \hat{I}(v; \varphi) \neq \emptyset \) and \( \Phi'_v(u) = 1 \). Since the variable set of \( \Phi'_v \) is \( \hat{I}(v; \varphi) \), we regard \( \Phi'_v \) as a CNF of \( |I(v; \varphi)| \) variables in this proof. For notational convenience, given a \( u \in \{0, 1\}^n \), we write \( \Phi'_v(u|_{I(v; \varphi)}) = 1 \) instead of \( \Phi'_v(u) = 1 \), where \( u|_S \) is the projection of \( u \) to \( S \subseteq \{1, 2, \ldots, n\} \). To check if there is a vector \( u|_{I(v; \varphi)} \neq (0, 0, \ldots, 0) \) such that \( \Phi'_v(u|_{I(v; \varphi)}) = 1 \), we consider two cases, \( \Phi'_v(0, 0, \ldots, 0) = 0 \) and \( \Phi'_v(0, 0, \ldots, 0) = 1 \). If \( \Phi'_v(0, 0, \ldots, 0) = 0 \), then solve H-SAT for \( \Phi'_v \) (see section 2); if the output of H-SAT is “yes” (resp., “no”), there is a desired vector \( u \) (resp., no desired vector \( u \)).
On the other hand, if $\Phi^*_v(0,0,\ldots,0) = 1$, then solve UNIQUE-H-SAT (see section 2) to see if $(0,0,\ldots,0)$ is the unique vector such that $\Phi^*_v(0,0,\ldots,0) = 1$. There is no desired vector $u$ if and only if $(0,0,\ldots,0)$ is the unique such vector. Since H-SAT and UNIQUE-H-SAT can be solved in time linear in the number of literals [7, 22], this can be done in $O(|\varphi|) = O(n|T|)$ time for each $v \in T$. Therefore, Step 3 can be executed in $O(n|T| \times |T|) = O(n|T|^2)$ time.

Summing up the time of all steps, we conclude that Algorithm CHECK-MINIMAL requires $O(|F||\varphi| + n|T||\varphi| + n|T|^2)$ time. \[\square\]

If a given DNF $\varphi$ is already a canonical Horn DNF, we can skip Steps 1 and 2 of CHECK-MINIMAL, leading to the following corollary.

**Corollary 5.1.** Given a pdBf $(T, F)$ and a canonical Horn DNF $\varphi$ of $(T, F)$, Problem MINIMAL-H-EXTENSION can be solved in $O(n|T|^2)$ time.

**Example 3.1:** Consider the pdBf $(T, F)$ and canonical Horn DNF $\varphi = \varphi^{(1)}$ of Example 3.1, to which we apply Algorithm CHECK-MINIMAL. Sets $I(v^{(i)}; \varphi)$, $i = 1, 2, \ldots, |T|$, are listed in Example 5.1. It can be seen that $\Phi^*_{v^{(i)}} = \perp$ for $i = 1, 2, 3, 4, 5$, $\Phi^*_{v^{(6)}} = \Phi^*_{v^{(7)}} = T$, and $\Phi^*_{v^{(8)}} = \emptyset$. Clearly, $\Phi^*_{v^{(i)}}$, $i = 6, 7, 8$, has a vector $u$ satisfying the condition in Step 3 of CHECK-MINIMAL. Consequently, this $\varphi$ does not represent a minimal Horn extension of $(T, F)$.

#### 5.2. Generating a minimal Horn extension.

In this subsection, we consider the generation of a minimal canonical Horn DNF of a given pdBf $(T, F)$. To generate a minimal canonical Horn DNF of a given pdBf $(T, F)$, we first construct a canonical Horn DNF $\varphi$ and then recursively check if some $\Phi^*_v$, $v \in T$, has a vector $u \in \{0, 1\}^n$ satisfying $ON(u) \cap I(v; \varphi) \neq \emptyset$, $\Phi^*_v(u) = 1$, and (5.13). If no, output $\varphi$ and halt. Otherwise, update $\varphi$ to a canonical Horn DNF $\varphi'$ such that $\varphi'(u) = 0$ and $\varphi' < \varphi$.

Note that condition (5.13) for $u$ is not restrictive, because $\Phi^*_v$ consists of only variables $x_j$ satisfying $j \in I(v; \varphi)$. Furthermore, by this restriction, $\Phi^*_v(u) = 1$ implies $\Phi^*_v(u) = 1$, and hence we can construct the above $\varphi'$. Formally, it can be written as follows.

**Algorithm FIND-MINIMAL**

Input: A pdBf $(T, F)$.

Output: A minimal canonical Horn DNF $\varphi$ of $(T, F)$ if $(T, F)$ has a Horn extension; otherwise, “no.”

**Step 1:** If $(T, F)$ has a Horn extension, construct a canonical DNF $\varphi = \bigvee_{v \in T} t_v$; otherwise, output “no” and halt.

**Step 2:** For each $v \in T$, check if $\Phi^*_v$ has a vector $u \in \{0, 1\}^n$ satisfying $ON(u) \cap I(v; \varphi) \neq \emptyset$, $\Phi^*_v(u) = 1$, and (5.13). If no $\Phi^*_v$ has such a vector $u$, then output the current $\varphi$ and halt. On the other hand, if $\Phi^*_v$ has such a vector $u$, based on this $u$, define

$$R_u(a; \varphi) = \{t \in R(a; \varphi) \mid t(u) = 0\}, \quad a \in T,$$

and reconstruct a canonical Horn DNF $\varphi$ by

$$\varphi := \bigvee_{a \in T} t_a, \quad t_a \in R_u(a; \varphi),$$

where $t_a \in R_u(a; \varphi)$ is chosen arbitrarily if $|R_u(a; \varphi)| \geq 2$. Return to Step 2.

In Step 2, if we have a desired vector $u$, $R_u(a; \varphi)$ of (5.14) can be easily obtained as follows:

$$R_u(a; \varphi) = \left\{ \begin{array}{ll}
R(a; \varphi) & \text{if } ON(a) \cap OFF(u) \neq \emptyset, \\
\{(\bigwedge_{l \in ON(a)} x_l) \bar{x}_l \mid l \in ON(u) \cap I(a; \varphi)\} & \text{otherwise.}
\end{array} \right.$$
THEOREM 5.2. Given a pdBf \((T, F)\), where \(T, F \subseteq \{0, 1\}^n\), a minimal canonical Horn DNF \(\varphi\) of \((T, F)\) can be generated in \(O(n|T|(|F| + n|T|^2))\) time if \((T, F)\) has a Horn extension.

Proof. FIND-MINIMAL is similar to CHECK-MINIMAL of subsection 5.1. To show its correctness, we need only prove that (i) \(R_u(a; \varphi) \neq \emptyset\) holds for all \(a \in T\) in (5.14) and (ii) FIND-MINIMAL will eventually halt.

(i) By the definition of \(\Phi^*_u\) and the assumption (5.13) on \(u\), \(\Phi^*_u(u) = 1\) implies \(\Phi_x(u) = 1\). This means that \(C_a(u) = 1\) holds for all \(a \in T\), where \(C_a = (\bigvee_{j \in \cal{ON}(u)} x_j)\) and hence some \(t \in R(a; \varphi)\) satisfies \(t(u) = 0\), by the definition of \(R_u(a; \varphi)\). Therefore, \(R_u(a; \varphi) \neq \emptyset\) holds for all \(a \in T\).

(ii) Let \(\varphi'\) be the DNF constructed by (5.15) from \(\varphi\) during an iteration. Then \(\varphi'\) is clearly a canonical Horn DNF of \((T, F)\), and \(f_{\varphi'} < f_\varphi\) holds since \(u \in T(\varphi) \setminus T(\varphi')\). More precisely,

\[
\sum_{a \in T} |R(a; \varphi')| < \sum_{a \in T} |R(a; \varphi)| \quad (\leq n|T|)
\]

holds. This is because \(R(a; \varphi') \subseteq R(a; \varphi)\) holds for all \(a \in T\), and \(t_v\) in \(\varphi\) is included in \(R(v; \varphi)\) but not in \(R(v; \varphi')\) (since \(\Phi^*_u(u) = 1\) implies \(t_v(u) = 1\)). Thus the number of iterations is at most \(n|T|\), which proves (ii).

Finally, let us consider its time complexity. By Theorem 3.1, Step 1 can be done in \(O(n|T||F|)\) time. In Step 2, for each \(v \in T\), we can obtain \(\Phi^*_u\) from \(\varphi\) in \(O(|\varphi|) = O(n|T|)\) time, since \(I(v; \varphi)\) can be computed in \(O(|\varphi|) = O(n|T|)\) time by applying F-CHAINING of subsection 5.1 to a vector \(v\). Similar to the proof of Theorem 5.1, for each \(v \in T\), a vector \(u\) satisfying \(\cal{ON}(u) \cap I(v; \varphi) \neq \emptyset\), \(\Phi^*_u(u) = 1\), and (5.13) can be computed in \(O(|T|)\) time, if there is at least one such \(u\). Thus \(O(n|T|)|T| = O(n|T|^2)\) time is required to find the desired vector \(u\). Once we have such a vector \(u\), \(t_u \in R_u(a; \varphi)\) can be obtained in \(O(n)\) time for each \(a \in T\). (See the discussion below Algorithm FIND-MINIMAL.) This means that the new \(\varphi\) of (5.15) can be obtained in \(O(n|T|)\) time. Hence each iteration of Step 2 requires \(O(n|T|^2)\) time. Since the number of iterations is at most \(n|T|\), Step 2 requires \(O(n^2|T|^3)\) time in total.

Consequently, Algorithm FIND-MINIMAL can be executed in \(O(n|T|(|F| + n|T|^2))\) time.

Example 5.4. Consider again the pdBf \((T, F)\) of Example 3.1. We demonstrate Algorithm FIND-MINIMAL by assuming that a canonical DNF \(\varphi = \varphi^{(1)}\) of Example 3.1 is obtained in Step 1.

First iteration. Sets \(I(v^{(i)}; \varphi)\), \(i = 1, 2, \ldots, 8\), are listed in Example 5.1. We have \(\Phi^*_{v^{(i)}} = \perp\) for \(i = 1, 2, \ldots, 5\), \(\Phi^*_{v^{(6)}} = \Phi^*_{v^{(7)}} = \top\), and \(\Phi^*_{v^{(8)}} = 8\), which are obtained in Example 5.3. For example, \(\Phi^*_{v^{(6)}}\) has a vector \(u = (111000001)\) satisfying the condition in Step 2 of FIND-MINIMAL. Based on this \(u\), we construct \(R_u(v^{(i)}; \varphi)\) of (5.14): \(R_u(v^{(1)}; \varphi) = \{123478\}, R_u(v^{(2)}; \varphi) = \{123579\}, R_u(v^{(3)}; \varphi) = \{123678\}, R_u(v^{(4)}; \varphi) = \{137\}, R_u(v^{(5)}; \varphi) = \{137\}, R_u(v^{(6)}; \varphi) = \{1239\}, R_u(v^{(7)}; \varphi) = \{1239\}, and R_u(v^{(8)}; \varphi) = \{1234567, 1234568, 1234569\}. By (5.15), we obtain a canonical Horn DNF

\[
\varphi := 123478 \lor 123579 \lor 123678 \lor 137 \lor 137 \lor 1239 \lor 1239 \lor 1234567,
\]

if 1234567 is chosen from \(R_u(v^{(8)}; \varphi)\).
Therefore, Horn DNF ϕ = \( \{ 1 \} \), \( \{ 1 \} \), \( \{ 1 \} \), \( \{ 1 \} \), \( \{ 1 \} \), \( \{ 1 \} \), \( \{ 1 \} \), \( \{ 1 \} \). Then we have \( \Phi^*_v = \varnothing \) for \( v = 1, 2, \ldots, 7 \) and \( \Phi^*_v = \{ 8 \} \). \( \Phi^*_v \) has a vector \( u = (1111110001) \) satisfying the condition in Step 2 of FIND-MINIMAL. By (5.14), we construct \( \{ 8 \}, \{ 9 \}, \{ 137 \}, \{ 137 \}, \{ 1239 \}, \{ 1239 \}, \{ 1239 \} \). This will prove the theorem, since Problem VERTEX COVER is known to be NP-hard [11]. Let \( G = (V, E) \) be an undirected graph, where \( V = \{ 1, \ldots, n \} \). Let us define \( T, F \subseteq \{ 0, 1 \}^n \) as follows:

\[
T = \{ x^A | A = V \setminus \{ i, j \}, (i, j) \in E \} \quad \text{and} \quad F = \varnothing,
\]

where \( x^A \) denotes the characteristic vector of set \( A \subseteq V \) (i.e., \( x^A = 1 \) if \( j \in A \) and \( x^A = 0 \) if \( j \notin A \) ). As \( F = \varnothing \), this \( (T, F) \) obviously has a Horn extension. Let \( \varphi = \bigvee_{v \in T} t_v \) with \( t_v = (\bigwedge_{j \in F, x_j} x^{i_j}) \) be a canonical Horn DNF that represents a minimum Horn extension of \( (T, F) \). We claim that \( |T(f_\varphi)| = |E| + \tau(G) \), where \( \tau(G) \) denotes the cardinality of a minimum vertex cover of \( G \). This will prove the theorem, since Problem VERTEX COVER (i.e., computing \( \tau(G) \)) is known to be NP-hard [11] and \( |T(f_\varphi)| \) can be computed from such a \( \varphi \) in polynomial time.

To prove the theorem, we first show that \( |T(f_\varphi)| \geq |E| + \tau(G) \). Since \( \varphi \) is a canonical Horn DNF, we have \( P_v = ON(v) \) (where \( |ON(v)| = n - 2 \), \( l_v \in I(v) = OFF(v) \)), and \( T(t_v) = \{ v, x^{V \setminus l_v} \} \). Therefore, \( T(f_\varphi) = T \cup \{ x^{V \setminus l_v} | v \in T \} \). Since \( OFF(v) \) of each \( v \in T \) corresponds to an edge of \( G \), set \( \{ l_v | v \in T \} \) forms a vertex cover of \( G \). Therefore,

\[
|T(\varphi)| = |T \cup \{ x^{V \setminus l_v} | v \in T \}|
= |E| + |\{ l_v | v \in T \}|
\geq |E| + \tau(G).
\]
Conversely, let $W \subseteq V$ be a minimum vertex cover with $|W| = \tau(G)$. Then define a Horn DNF $\varphi_w = \bigvee_{v \in T} t_v$, where $t_v = (\bigwedge_{j \in ON(v)} x_j) \bar{x}_w$, for some $w \in OFF(v) \cap W$. Then $T(t_v) = \{v, x^{\bigwedge \{j\}}\}$ holds and $\varphi_w$ is a Horn DNF of $(T, F)$. Furthermore, $|T(\varphi_w)| = |T \cup \{x^{\bigwedge \{k\}} \mid k \in W\}| = |E| + |W| = |E| + \tau(G)$. 

However, the uniqueness of a minimal Horn extension can be decided in polynomial time.

**Problem UMIN-H-EXTENSION**

**Input:** A pdBf $(T, F)$.

**Question:** Does $(T, F)$ have the unique minimal Horn extension?

**Lemma 5.8.** Let $\varphi = \bigvee_{v \in T} t_v$ be a minimal canonical Horn DNF of a pdBf $(T, F)$. Then $(T, F)$ does not have the unique minimal Horn extension (which is $f_\varphi$) if and only if at least one of the CNFs

$$
(5.16) \quad \Phi^+_v = t_v \land \bigwedge_{a \in T} C^+_a, \quad v \in T, 
$$

where $C^+_a = (\bigvee_{j \in ON(a)} x_j \lor \bigvee_{j \in I(a)} x_j)$, is satisfiable.

**Proof.** To show the if-part, let us first assume that a vector $u$ satisfies $\Phi^+_v(1) = 1$. Then obviously $\varphi(u) = 1$ holds, and, for each $a \in T$, there is a term $t^*_a$ in $R(v)$ such that $t^*_a(u) = 0$. By choosing such a term $t^*_a$ for each $a \in T$, we have a canonical Horn DNF $\psi = \bigvee_{a \in T} t^*_a$ of $(T, F)$ such that $\psi(u) = 0$. Therefore, $u$ satisfies $\varphi(u) = 1$ and $\psi(u) = 0$, which proves that $(T, F)$ has at least two minimal Horn extensions.

Conversely, if no $\Phi^+_v$ is satisfiable, then, for every vector $u$ such that $\varphi(u) = 1$, some clause $C^+_a$ satisfies $C^+_a(u) = 0$. By the definition of $I(a)$, this implies that $t^*_a(u) = 1$ holds for all $a \in T$. Therefore, if a vector $u$ satisfies $\varphi(u) = 1$, then $\psi(u) = 1$ must hold for all canonical Horn DNFs $\psi = \bigvee_{a \in T} t^*_a$, which shows that $(T, F)$ has the unique minimal Horn extension. 

Note that this lemma corresponds to Lemma 5.3, and all other lemmas, theorems, and algorithms in subsection 5.1 are valid, even if $\Phi_v$ and $I(v; \varphi)$ are replaced by $\Phi^+_v$ and $I(v)$, respectively. (Recall that $I(v)$ becomes $I(v; \varphi)$ if $\varphi$ represents $f_{\max}$.) Therefore, in order to solve Problem UMIN-H-EXTENSION, first construct a canonical DNF $\varphi$ such that $f_\varphi$ is a minimal Horn extension by Algorithm FIND-MINIMAL in subsection 5.2, and then check the condition in Lemma 5.8 by using Algorithm CHECK-MINIMAL with the above replacement incorporated.

**Theorem 5.4.** Problem UMIN-H-EXTENSION can be solved in $O(n|T|(|F| + n|T|^2))$ time.

**Proof.** We consider only its time complexity. A minimal canonical Horn DNF $\varphi$ can be constructed in $O(n|T|(|F| + n|T|^2))$ time by Theorem 5.2. We also can check the condition in Lemma 5.8 in $O(n|T|^2)$ time (Corollary 5.1) by using the modified CHECK-MINIMAL.

**6. Shortest Horn extensions.** Finally, we show that the following problem, related to the knowledge compression for expert systems [13, 14], is intractable:

**Problem SHORTEST-H-EXTENSION**

**Input:** A pdBf $(T, F)$ and a positive integer $k$.

**Question:** Is there a Horn DNF $\varphi$ of $(T, F)$ such that $|\varphi| \leq k$?

**Theorem 6.1.** Problem SHORTEST-H-EXTENSION is NP-complete.

**Proof.** This problem is in NP, since we can check in polynomial time if a given DNF $\varphi$ represents a Horn extension of $(T, F)$ and satisfies $|\varphi| \leq k$. Now we transform
Problem VERTEX COVER to this problem, where VERTEX COVER is known to be NP-hard [11]. Let $G = (V, E)$ be an undirected graph, where $V = \{1, 2, \ldots, n\}$. Let us define $T, F \subseteq \{0, 1\}^n$ as follows:

\[
T = \{x^A \mid A = V \setminus \{i, j\}, (i, j) \in E\}, \\
F = \{e = (11 \ldots 1)\},
\]

where $x^A$ denotes the characteristic vector of set $A \subseteq V$.

We claim that there is a Horn DNF $\varphi$ of $(T, F)$ such that $|\varphi| \leq k$ if and only if $\tau(G) \leq k$, where $\tau(G)$ denotes the cardinality of a minimum vertex cover of $G$. Similar to the proof of Theorem 5.3, this will complete the proof.

Let $\varphi = \bigvee_{l \in L} t_l$, where $t_l = \bigcap_{j \in T} x_j \bigcap_{j \in N_l} \bar{x}_j$, be a Horn DNF of $(T, F)$ such that $|\varphi| \leq k$. Then, for every $l \in L$, the following conditions hold:

(a) $N_l \neq \emptyset$, i.e., $|N_l| = 1$ holds since otherwise $t_l(e) = 1$, which is a contradiction.

(b) $P_l = \emptyset$ holds, since replacing $P_l$ by $\emptyset$ produces a shorter good term $t_l$ such that $|t_l| \leq |t_l|$, and $t_l(a) = 1$ for all $a \in T$, and $t_l'(e) = 0$.

Furthermore, since $\varphi(a) = 1$ for every $a \in T$, there must exist an $l \in L$ such that $t_l(a) = 1$ (i.e., $N_l \subseteq \text{OFF}(a) = \{i, j\}$ for the corresponding edge $(i, j) \in E$). Hence $\bigcup_{l \in L} N_l$ is a vertex cover of $G$ such that $|\bigcup_{l \in L} N_l| = |\varphi| \leq k$.

Conversely, if $W$ is a vertex cover with $|W| \leq k$, then $\varphi_w = \bigvee_{j \in W} \bar{x}_j$ is a Horn DNF of $(T, F)$ such that $|\varphi_w| \leq k$. \(\square\)

7. Conclusion and future research. Because there are in general many Horn extensions for a given pdBf $(T, F)$, in this paper we considered Horn extensions with special properties. In particular, we investigated maximal and minimal Horn extensions and pointed out that the maximal Horn extension is always unique (i.e., maximum) but there are many minimal Horn extensions. The main contribution of this paper is to show that checking if a Horn DNF is minimal and generating a minimal Horn DNF of a pdBf $(T, F)$ both can be done in polynomial time. We can also check in polynomial time if a minimal extension is unique. However, the problems of finding a Horn DNF of a minimum Horn extension and finding a shortest Horn DNF of a pdBf $(T, F)$ are shown to be NP-hard.

A possible topic for future research is the development of an efficient algorithm to generate all minimal Horn extensions of a given pdBf $(T, F)$.

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