Graph Theory Uncovers the Roots of Perfection

A newly minted proof tells how to recognize which arrangements of points and lines are the crème de la crème

To some, perfection is priceless. But for four graph theorists, it has a very specific value. If their solution to one of the oldest problems in their discipline—a classification of so-called perfect graphs—holds up, they will reap a $10,000 bounty.

The strong perfect graph conjecture (SPGC) has perplexed mathematicians for more than 40 years. “It’s a problem that everyone in graph theory knows about, and some people in related areas, particularly linear programming,” says Paul Seymour of Princeton University, who announced the proof at a meeting of the Canadian Mathematical Society last month. Its solution might enable mathematicians to quickly identify perfect graphs, which have properties that make otherwise intractable problems involving networks easy to solve.

The graphs in question consist of nothing more than dots and lines. Each line connects exactly two dots, or nodes. The SPGC grew out of mathematicians’ fascination with coloring graphs in such a way that no two nodes of the same color are connected, a problem rooted in the real-world business of coloring maps. When Wolfgang Haken and Kenneth Appel proved the famous Four-Color Theorem for planar maps in 1976, they did it by means of graph theory.

Coloring problems make sense for other kinds of graphs as well. In a cell-phone network, for example, the nodes are transmitters, the lines connect any two transmitters whose ranges overlap, and the colors correspond to channels. Coloring the network amounts to assigning channels so that no adjacent transmitters broadcast on the same channel. Of course, the phone company would want to use the smallest possible number of channels, which is called the chromatic number chi (\(\chi\)) of the network.

It’s easy to see that any group of nodes that are all connected to one another must all be different colors. Graph theorists call such a dense web of nodes a clique. Thus, in any graph, chi has to be at least as large as the size of the biggest clique, a number known as omega (\(\omega\)). In a perfect graph, in fact, chi equals the largest number of interconnected nodes (omega). Adding two transmitters that create an “odd hole” (arrows) makes the graph imperfect.

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Dana Mackenzie is a writer in Santa Cruz, California.