Group - A group is a pair \( \langle G, * \rangle \) with

- A binary operation * defined on a set G

with the following properties:

(G-1) **Associativity**

\[ a * (b * c) = (a * b) * c \]

(G-2) **Identity**

\[ \exists \text{ an element } e \in G \text{ s.t. } \]

\[ e * a = a * e = a \quad \forall a \in G \]

(G-3) **Inverse**

\[ \text{for } a \in G, \text{ there exists } a' \in G \text{ s.t. } \]

\[ a * a' = a' * a = e \]

Given a set \( X \), a binary operation * assigns

each element \( a \) of \( X \) to every pair

of elements \( \langle a, b \rangle \) of \( X \). This element assigns

an element of \( X \).
Ex.:

1) \( \langle \mathbb{R}, + \rangle \)
   - **group** under **addition**
   - **0** is **neutral**
   - **-1** is **inverse**

2) \( \langle \mathbb{R}, + \rangle \)
   - **addition**
   - **0** is **neutral**
   - **-1** is **inverse**

3) \( \langle \mathbb{R}^+, \times \rangle \)
   - **positive** real numbers
   - **multiplication**

4) **set of** \( M \times N \) **real numbers**
   - **matrix addition**

5) **set of** all \( M \times N \) **real numbers**
   - **matrix multiplication**
Field  A field \( F \) is a triple \((F, +, \cdot)\), where + and \(\cdot\) are two binary operators defined on the set \( F \) such that \( a \times 0 = F - I, F - II, F - III \) below are satisfied. The elements of a field are called scalars, denoted by \( \alpha, \beta, \gamma \ldots \).

\((F-1)\) For every pair of scalars \( \alpha, \beta \), we have defined \( \alpha + \beta \) as the "sum" of \( \alpha, \beta \) with properties:

So, \( F \) is a group with + as a binary operator.

\( \begin{cases} 
(1) & \alpha + \beta = \beta + \alpha \quad \text{commutativity} \\
(2) & (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \quad \text{associative} \\
(3) & \exists \text{ unique scalars } 0 \in F, \quad 0 + \alpha = \alpha + 0 = \alpha \\
(4) & \forall \alpha \in F, \exists \text{ unique scalars } (-\alpha) \in F \quad \text{such that} \\
& \alpha + (-\alpha) = 0 
\end{cases} \)
The scalar $\alpha \cdot \beta$ is called the product of $\alpha$ & $\beta$ and has the following properties:

1. $\alpha \cdot \beta = \beta \cdot \alpha$ - commutative
2. $(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot \gamma) \cdot \beta$ - associative
3. identity - exists only scalar 1 not.
   $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$ - identity element
4. $\forall \alpha \neq 0 \exists \alpha^{-1}$ scalar $\alpha^{-1}$. $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$. \\

   (See unique scalar of additive identity of field (F-1) (3)).

The sum and the product obey the distributive property:

$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
Exs. of Fields:

1) rational nrs \( \mathbb{Q} \) with \( + \).

2) \( \mathbb{R} \) with \( + \).

3) \( \mathbb{C} \) with \( + \).

4) \( \mathbb{Z}^+ \) not a field (no inverse for \( \cdot \)).
   \( \mathbb{Z} \) with \( + \cdot \) is not a field (no inverse for \( \cdot \)).

Vector Space:

Let \( F \) be a field and \( + \) be a binary op. on a set \( V \). The triple \( (V, +, F) \) is a vector space over a field \( F \) if the following properties are satisfied:

See next page.
(V-1) If \( f, g \in V \) then exist a common vector \( f + g \in V \) called the sum of \( f \) and \( g \).

\[
\begin{align*}
(1) \quad f + g &= g + f \\
(2) \quad f + (g + h) &= (f + g) + h \\
(3) \quad \exists \text{ zero vector } 0 \in V \text{ s.t. } f + 0 = f \\
(4) \quad \text{To every } f \in V \text{ exist a vector } -f \in V \text{ s.t. } f + (-f) = 0 \\
\end{align*}
\]

(V-2) For every \( \alpha \in \mathbb{F} \) and \( f \in V \) there exist a \( \alpha f \in V \), called \( \alpha \)-multiplication, s.t.

\[
\alpha (f + g) = (\alpha f) + (\alpha g)
\]

\[
\alpha (-f) = - (\alpha f) = (\alpha f) - f
\]

The linear space \( V \) is a field. 

Identify \( \mathbb{F} \) with zero vector of \( V \).
Let $V$ be a vector space over $\mathbb{F}$. Given $\alpha \in \mathbb{F}, \forall f, g \in V$ we have:

\[ \alpha (f + g) = \alpha f + \alpha g \]

This operation is an algebra over $\mathbb{F}$. For specific vectors $\mathbf{u}$, $\mathbf{v}$, the operation $\mathbf{u} \cdot \mathbf{v}$ (not just matrix multiplication) is denoted by $\mathbf{u} \cdot \mathbf{v}$.

For any vector $\mathbf{v}$, $\mathbf{0}$ is the zero vector in $V$.

\[ \alpha (f + g) = \alpha f + \alpha g \]

Axiom: For any $\alpha \in \mathbb{F}$, $\mathbf{0} \cdot \mathbf{f} = \mathbf{0}$ for any $\mathbf{f} \in V$ if and only if $\mathbf{0} \cdot (\alpha f + \alpha g) = \alpha \mathbf{0} f + \alpha \mathbf{0} g$. 

Ex. 3

(1) Every field \( F \) is also a vector space over \( F \) as a field of scalars.

E.g. Set of all real numbers \( \mathbb{R} \) is a real vector space with \( \mathbb{R} \) as a field of scalars.

(2) Set of all \( n \)-tuples \( (x_1, x_2, \ldots, x_n) \) where \( x_k \in F \) is a vector space over \( F \).

This set is a vector space with \( F \) as field of scalars.

E.g.

(2-1) \( \mathbb{C}^n \) is a complex vector space with \( \mathbb{C} \) as the field of scalars.

(2-2) \( \mathbb{R}^n \) is a real vector space with \( \mathbb{R} \) as the field of scalars.

(2-3) \( \mathbb{Q}^n \) is a rational vector space with \( \mathbb{Q} \) as the field of scalars.

TO BE CONTINUED