Phantom Types and Subtyping*

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Abstract

We investigate a technique from the literature, called the phantom types technique, that uses parametric polymorphism, type constraints, and unification of polymorphic types to model a subtyping hierarchy. Hindley-Milner type systems, such as the one found in SML, can be used to enforce the subtyping relation. We show that this technique can be used to encode any finite subtyping hierarchy (including hierarchies arising from multiple interface inheritance). We formally demonstrate the suitability of the phantom types technique for capturing subtyping by exhibiting a type-preserving translation from a simple calculus with bounded polymorphism to a calculus embodying the type system of SML. We then illustrate a particular use of the technique to capture programming invariants associated with user-defined datatypes, in the form of statically-enforced datatype specializations.

1 Introduction

It is well known that traditional type systems, such as the one found in Standard ML [Milner, Tofte, Harper, and MacQueen 1997], with parametric polymorphism and type constructors can be used to capture program properties beyond those naturally associated with a Hindley-Milner type system [Milner 1978]. For concreteness, let us review a simple example, due to Leijen and Meijer [1999]. Consider a type of atoms, either booleans or integers, that can be easily represented as an algebraic datatype:

```
datatype atom = I of int | B of bool
```

There are a number of operations that we may perform on such atoms (see Figure 1(a)). When the domain of an operation is restricted to only one kind of atom, as with `conj` and `double`, a run-time check must be made and an error or exception reported if the check fails.

One aim of static type checking is to reduce the number of run-time checks by catching type errors at compile time. Of course, in the example above, the SML type system does not consider `conj (mkI 3, mkB true)` to be ill-typed; evaluating this expression will simply raise a run-time exception.

If we were working in a language with subtyping, we would like to consider integer atoms and boolean atoms as distinct subtypes of the general type of atoms and use these subtypes to refine the

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```
fun mkI (i:int):atom = I (i)
fun mkB (b:bool):atom = B (b)
fun toString (v:atom):string =
  (case v
    of I (i) => Int.toString (i)
     | B (b) => Bool.toString (b))
fun double (v:atom):atom =
  (case v
    of I (i) => I (i * 2)
     | _ => raise Fail "type mismatch")
fun conj (v1:atom,
          v2:atom):atom =
  (case (v1,v2)
    of (B (b1), B (b2)) => B (b1 andalso b2)
     | _ => raise Fail "type mismatch")
```

(a) Unsafe operations

```
fun mkI (i:int):int atom = I (i)
fun mkB (b:bool):bool atom = B (b)
fun toString (v:'a atom):string =
  (case v
    of I (i) => Int.toString (i)
     | B (b) => Bool.toString (b))
fun double (v:int atom):int atom =
  (case v
    of I (i) => I (i * 2)
     | _ => raise Fail "type mismatch")
fun conj (v1:bool atom,
          v2:bool atom):bool atom =
  (case (v1,v2)
    of (B (b1), B (b2)) => B (b1 andalso b2)
     | _ => raise Fail "type mismatch")
```

(b) Safe operations

Figure 1:

types of the operations. Then the type system would report a type error in the expression `double (mkB false)` at compile time. Fortunately, we can write the operations in a way that utilizes the SML type system to do just this. We change the definition of the datatype to the following:

```
datatype 'a atom = I of int | B of bool
```

and constrain the types of the operations (see Figure 1(b)). We use the superfluous type variable in the datatype definition to encode information about the kind of atom. (Because instantiations of this type variable do not contribute to the run-time representation of atoms, it is called a *phantom type*.) The type `int atom` is used to represent integer atoms and `bool atom` is used to represent boolean atoms. Now, the expression `conj (mkI 3, mkB true)` results in a compile-time type error, because the types `int atom` and `bool atom` do not unify. (Observe that our use of `int` and `bool` as phantom types is arbitrary; we could have used any two types that do not unify to make the integer versus boolean distinction.) On the other hand, both `toString (mkI 3)` and `toString (mkB true)` are well-typed; `toString` can be used on any atom. This is the essence of the technique explored in this paper: using a free type variable to encode subtyping information and using an SML-like type system to enforce the subtyping. This "phantom types" technique, where user-defined restrictions are reflected in the constrained types of values and functions, underlies many interesting uses of type systems. It has been used to derive early implementations of extensible records [Wand 1987; Rémy 1989; Burton 1990], to provide a safe and flexible interface to the Network Socket API [Reppy 1996], to interface to COM components [Finne, Leijen, Meijer, and Peyton Jones 1999], to type embedded compiler expressions [Leijen and Meijer 1999; Elliott, Finne, and de Moor 2000], to record sets of effects in type-and-effect type systems [Pessaux and Leroy 1999], and to embed a representation of the C type system in SML [Blume 2001].

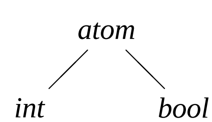
The first contribution of this paper is to describe a general procedure for applying the phantom types technique to subtyping. This procedure relies on an appropriate encoding of the subtyping hierarchy. We exhibit different classes of encodings for different kinds of hierarchies. The second contribution of this paper is to formalize this use of phantom types and prove its correctness. We present a type-preserving translation from a calculus with subtyping to a calculus with let-bounded polymorphism, using the procedure described earlier. The kind of subtyping that can be captured turns out to be an interesting variant of bounded polymorphism [Cardelli, Martini, Mitchell, and Scedrov 1994], with a very restricted subsumption rule.

As a nontrivial application of the phantom types technique to encode subtyping hierarchies, we show how to capture programming invariants associated with user-defined datatypes. Dubbed datatype specialization, this technique uses type-checking to verify that program invariants are preserved; compile-time type errors will indicate errors that could violate program invariants.

This paper is structured as follows. In the next section, we describe a simple recipe for deriving an interface enforcing a given subtyping hierarchy. The interface is parameterized by an encoding, via phantom types, of the subtyping hierarchy. In Section 3, we examine different encodings for hierarchies. We also address the issue of extensibility of the encodings. In Section 4, we extend the recipe to capture a limited form of bounded polymorphism. In Section 5, we formally define the kind of subtyping captured by our encodings by giving a simple calculus with subtyping and showing that our encodings provide a type-preserving translation to a variant of the Damas-Milner calculus, embodying the essence of the SML type system. In Section 6, we examine how to encode datatype specialization using phantom types, following the approach to capture subtyping we have described. We conclude with some problems inherent to the approach and a consideration of future work. The formal details of the calculi we introduce in Section 5, the proofs of our results, as well as other supplemental material can be found in the appendices.

2 From subtyping to polymorphism

The example in the introduction has the following features: an underlying primitive type of values (the original type `atom`), a set of operations, and "implicit" subtypes that correspond to the sensible domains of the operations. The subtyping hierarchy corresponding to the example is as follows:



The subtyping hierarchy is modeled by assigning a type to every implicit subtype in the hierarchy. For instance, integer atoms with implicit subtype *int* are encoded by the SML type `int atom`. The appropriate use of polymorphic type variables in the type of an operation indicates the maximal type in the domain of the operation. For instance, the operation `toString` has the conceptual type *atom* → *string* which is encoded by the SML type `'a atom → string`. The key observation is the use of type unification to enforce the subtyping hierarchy: an `int atom` can be passed to a function expecting an `'a atom`, because these types unify.

We consider the following problem. Given an abstract primitive type τ_p, a subtyping hierarchy, and an implementation of τ_p and its operations, we wish to derive a "safe" SML signature which uses phantom types to encode the subtyping and a "safe" implementation from the "unsafe" implementation. We will call the elements of the subtyping hierarchy *implicit types* and talk about *implicit subtyping* in the hierarchy. All values share the same underlying representation and each operation has a single implementation that acts on this underlying representation. The imposed subtyping captures restrictions that arise because of some external knowledge about the semantics of the operations; intuitively, it captures a "real" subtyping relationship that is not exposed by the abstract type.

We first consider deriving the safe interface. The new interface defines a type $\alpha\ \tau$ corresponding to the abstract type τ_p. The type variable α will be used to encode implicit subtype information. We require an encoding $\langle\sigma\rangle$ of each implicit type σ in the hierarchy; this encoding should yield a type in the underlying SML type system, with the property that $\langle\sigma_1\rangle$ unifies with $\langle\sigma_2\rangle$ if and only if σ_1 is an implicit subtype of σ_2. An obvious issue is that we want to use unification (a symmetric relation) to capture subtyping (an asymmetric relation). The simplest approach is to use two encodings $\langle\cdot\rangle_C$ and $\langle\cdot\rangle_A$ defined over all the implicit types in the hierarchy. A *value* of implicit type σ will be assigned a type $\langle\sigma\rangle_C\ \tau$. We call $\langle\sigma\rangle_C$ the *concrete* subtype encoding of σ, and we assume that it uses only ground types (i.e., no type variables). In order to restrict the domain of an operation to the set of values in any implicit subtype of σ, we use $\langle\sigma\rangle_A$, the *abstract* subtype encoding of σ. In order for the underlying type system to enforce the subtype hierarchy, we require the encodings $\langle\cdot\rangle_C$ and $\langle\cdot\rangle_A$ to be *respectful* by satisfying the following property:

$$\text{for all } \sigma_1 \text{ and } \sigma_2,\ \langle\sigma_1\rangle_C \text{ matches } \langle\sigma_2\rangle_A \text{ iff } \sigma_1 \leq \sigma_2.$$

For example, the encodings used in the introduction are respectful:

$$\begin{array}{rclrcl} \langle \textit{atom}\rangle_A & = & \texttt{'a atom} & \langle \textit{atom}\rangle_C & = & \texttt{unit atom} \\ \langle \textit{int}\rangle_A & = & \texttt{int atom} & \langle \textit{int}\rangle_C & = & \texttt{int atom} \\ \langle \textit{bool}\rangle_A & = & \texttt{bool atom} & \langle \textit{bool}\rangle_C & = & \texttt{bool atom} \end{array}$$

The utility of the phantom types technique relies on being able to find respectful encodings for subtyping hierarchies of interest.

To allow for matching, the abstract subtype encoding will introduce free type variables. Since, in a Hindley-Milner type system, a type cannot contain free type variables, the abstract encoding will be part of the larger type scheme of some polymorphic function operating on the value of implicit subtypes. This leads to some restrictions on when we should constrain values by concrete or abstract encodings. For the time being, we will restrict ourselves to using concrete encodings in all covariant type positions, and using abstract encodings in most contravariant type positions. We will return to this issue in Section 4. Another consequence of having the abstract encoding be part of a larger type scheme that binds the free variables in prenex position is that the subtyping is resolved not at the point of function application, but rather at the point of *type application*, when the type variables are instantiated. We postpone a discussion of this important point to Section 4, where we extend our recipe to account for a form of bounded polymorphism.

Consider again the example from the introduction. Assume we have encodings $\langle\cdot\rangle_C$ and $\langle\cdot\rangle_A$ for the hierarchy and a structure `Atom` implementing the "unsafe" operations, with the signature given in Figure 2(a). Deriving an interface using the recipe above, we get the safe signature given in Figure 2(b).

We must now derive a corresponding "safe" implementation. We need a type $\alpha\ \tau$ isomorphic to τ_p such that the type system considers $\tau_1\ \tau$ and $\tau_2\ \tau$ equivalent iff τ_1 and τ_2 are equivalent. (Note that this requirement precludes the use of type abbreviations of the form `type` $\alpha\ \tau$ `=` τ_p, which define constant type functions.) We can then constrain the types of values and operations using $\langle\sigma\rangle_C\ \tau$ and $\langle\sigma\rangle_A\ \tau$. In SML, the easiest way to achieve this is to use an abstract type at the module system level, as shown in Figure 3(a). The use of an opaque signature is critical to get the required behavior in terms of type equivalence. The advantage of this method is that there is no overhead.

```
signature ATOM = sig
  type atom
  val int : int -> atom
  val bool : bool -> atom
  val toString : atom -> string
  val double : atom -> atom
  val conj : atom * atom -> atom
end
```

(a) Unsafe signature

```
signature SAFE_ATOM = sig
  type 'a atom
  val int : int -> ⟨int⟩_C atom
  val bool : bool -> ⟨bool⟩_C atom
  val toString : ⟨top⟩_A atom -> string
  val double : ⟨int⟩_A atom -> ⟨int⟩_C atom
  val conj : ⟨bool⟩_A atom * ⟨bool⟩_A atom -> ⟨bool⟩_C atom
end
```

(b) Safe signature

Figure 2:

```
structure SafeAtom1 :> SAFE_ATOM = struct
  type 'a atom = Atom.atom
  val int = Atom.int
  val bool = Atom.bool
  val toString = Atom.toString
  val double = Atom.double
  val conj = Atom.conj
end
```

(a) Opaque signature

```
structure SafeAtom2 : SAFE_ATOM = struct
  datatype 'a atom = C of Atom.atom
  fun int (i) = C (Atom.int (i))
  fun bool (b) = C (Atom.bool (b))
  fun toString (C v) = Atom.toString (v)
  fun double (C v) = C (Atom.double (v))
  fun conj (C b1, C b2) = C (Atom.conj (b1,b2))
end
```

(b) Datatype declaration

Figure 3:

In a language without abstract types at the module level, another approach is to wrap the primitive type τ_p using a datatype declaration

$$\texttt{datatype 'a } \tau \texttt{ = C of } \tau_p$$

The type $\alpha\ \tau$ behaves as required, because the datatype declaration defines a generative type operator. However, we must explicitly convert primitive values to and from $\alpha\ \tau$ to witness the isomorphism. This yields the implementation given in Figure 3(b).

We should stress that the "safe" interface must ensure that the type $\alpha\ \tau$ is abstract—either through the use of opaque signature matching, or by hiding the value constructors of the type. Otherwise, it may be possible to create values that do not respect the subtyping invariants enforced by the encodings. Similarly, the use of an abstract subtype encoding in a covariant type position can lead to violations in the subtyping invariants.

We now have a way to derive a safe interface and implementation, by adding type information to a generic, unsafe implementation. In the next section, we show how to construct respectful encodings $\langle\cdot\rangle_C$ and $\langle\cdot\rangle_A$ by taking advantage of the structure of the subtyping hierarchy.

3 Encoding subtyping hierarchies

The framework presented in the previous section relies on having concrete and abstract encodings of the implicit subtypes in the subtyping hierarchy with the property that unification of the results of the encoding respects the subtype relation. In this section, we describe how such encodings can be obtained. Different encodings are appropriate, depending on the characteristics of the subtyping hierarchy being encoded. Most of these encodings assume that the subtyping relation is completely known *a priori*. We address the question of extensibility in Section 3.5.

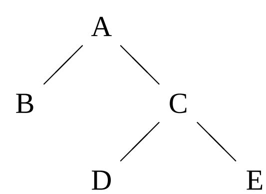
3.1 Tree hierarchies

For the time being, we restrict ourselves to finite subtyping hierarchies. The simplest case to handle is a tree hierarchy. Since this is the type of hierarchy that occurs in the example we saw in the introduction (and, in fact, in all the examples we found in the literature on encoding subtyping hierarchies in a polymorphic type system), this encoding should be clear. The idea is to assign a type constructor to every subtype of a subtyping hierarchy. Assume we have an encoding $\langle\cdot\rangle_N$ assigning a distinct (syntactic) name to each entry in a subtyping hierarchy H. Hence, for each $\sigma \in H$, we define:

$$\texttt{datatype 'a } \langle\sigma\rangle_N \texttt{ = C\_}\langle\sigma\rangle_N$$

(The name of the data constructor is completely irrelevant, as we will never construct values of this type.[1])

For example, consider the following subtyping hierarchy (which is essentially the one used in the Sockets API described by Reppy [1996]):



We first define type constructors for every element of the hierarchy. We assume a reasonable name encoding $\langle\cdot\rangle_N$, such as $\langle A\rangle_N = \texttt{LA}$, $\langle B\rangle_N = \texttt{LB}$, etc. Hence, we have

```
datatype 'a LA = C_LA
```

and likewise for the other elements. The concrete encoding for an element of the hierarchy represents the path from the top of the hierarchy to the element itself. Hence,

$$\begin{array}{lcl}
\langle A\rangle_C & = & \texttt{unit LA} \\
\langle B\rangle_C & = & \texttt{(unit LB) LA} \\
\langle C\rangle_C & = & \texttt{(unit LC) LA} \\
\langle D\rangle_C & = & \texttt{((unit LD) LC) LA} \\
\langle E\rangle_C & = & \texttt{((unit LE) LC) LA}
\end{array}$$

For the corresponding abstract encoding to be respectful, we require the abstract encoding of σ to unify with the concrete encoding of all the subtypes of σ. In other words, we equire the abstract

[1] Again, we could use an abstract type via the SML module system instead of a datatype.

encoding to represent the prefix of the path leading to the element σ in the hierarchy. We use a type variable to unify with any part of the path after the prefix we want to represent. Hence,

$$\begin{aligned}
\langle A\rangle_A &= \alpha_1\ \texttt{LA} \\
\langle B\rangle_A &= (\alpha_2\ \texttt{LB})\ \texttt{LA} \\
\langle C\rangle_A &= (\alpha_3\ \texttt{LC})\ \texttt{LA} \\
\langle D\rangle_A &= ((\alpha_4\ \texttt{LD})\ \texttt{LC})\ \texttt{LA} \\
\langle E\rangle_A &= ((\alpha_5\ \texttt{LE})\ \texttt{LC})\ \texttt{LA}
\end{aligned}$$

We can then verify, for example, that the concrete encoding of D unifies with the abstract encoding of C, as required.

Note that $\langle\cdot\rangle_A$ requires every type variable α_i to be a fresh variable, unique in its context. This ensures that we do not inadvertently refer to any type variable bound in the context where we are introducing the abstractly encoded type. The following example illustrates the potential problem. Let H be the subtyping hierarchy given above, over some underlying type τ_p. Suppose we wish to encode an operation $f : A \times A \rightarrow$ *int* in a way that it can accept any implicit subtype of A for its two arguments. The encoded type of the operation becomes $f : \langle A\rangle_A\ \tau \times \langle A\rangle_A\ \tau \rightarrow \texttt{int}$ (where $\alpha\ \tau$ is the wrapped type of τ_p values) which should translate to $f : (\texttt{'a LA})\ \tau \times (\texttt{'b LA})\ \tau \rightarrow \texttt{int}$. If we are not careful in choosing fresh type variables, we could generate the following type $f : (\texttt{'a LA})\ \tau \times (\texttt{'a LA})\ \tau \rightarrow \texttt{int}$, corresponding to a function that requires two arguments of the same type, which is not the intended meaning. (The handling of introduced type variables is somewhat delicate; we address the issue in more detail in Section 4.)

It should be clear how to generalize the above discussion to concrete and abstract encodings for arbitrary finite tree hierarchies. Let $\top_H$ correspond to the root of the finite tree hierarchy. Define an auxiliary encoding $\langle\cdot\rangle_{\mathcal{X}}$ which can be used to construct chains of type constructors:

$$\begin{aligned}
\langle\top_H\rangle_{\mathcal{X}}\ t &= t\ \langle\top_H\rangle_N \\
\langle\sigma\rangle_{\mathcal{X}}\ t &= \langle\sigma'\rangle_{\mathcal{X}}\ (t\ \langle\sigma\rangle_N) \quad \text{if } \sigma' \text{ is the parent of } \sigma
\end{aligned}$$

Using this auxiliary encoding, we can define the concrete and abstract encodings:

$$\begin{aligned}
\langle\sigma\rangle_C &= \langle\sigma\rangle_{\mathcal{X}}\ \texttt{unit} \\
\langle\sigma\rangle_A &= \langle\sigma\rangle_{\mathcal{X}}\ \alpha \qquad \alpha \text{ fresh}
\end{aligned}$$

3.2 Finite powerset lattices

Not every subtyping hierarchy of interest is a tree. More general hierarchies can be used to model multiple interface inheritance in an object-oriented setting. Let us now examine more general subtyping hierarchies. We first consider a particular lattice that will be useful in our development. Recall that a lattice is a hierarchy where every set of elements has both a least upper bound and a greatest lower bound. Given a finite set S, we let the *powerset lattice* of S be the lattice of all subsets of S, ordered by inclusion, written $(\wp(S), \subseteq)$. We now exhibit an encoding of powerset lattices.

Let n be the cardinality of S and assume an ordering $s_1, \ldots, s_n$ on the elements of S. We encode subset X of S as an n-tuple type, where the i^{th} entry expresses that $s_i \in X$ or $s_i \notin X$.

First, we introduce a datatype definition:

```
datatype 'a z = Z
```

(The name of the datatype constructor is irrelevant, because we will never construct values of this type.) The encoding of an arbitrary subset of S is given by:

$$\langle X\rangle_C = t_1 \times \ldots \times t_n \quad \text{where } t_i = \begin{cases} \texttt{unit} & \text{if } s_i \in X \\ \texttt{unit z} & \text{otherwise} \end{cases}$$

$$\langle X\rangle_A = t_1 \times \ldots \times t_n \quad \text{where } t_i = \begin{cases} \alpha_i & \text{if } s_i \in X \\ \alpha_i \ \texttt{z} & \text{otherwise} \end{cases}$$

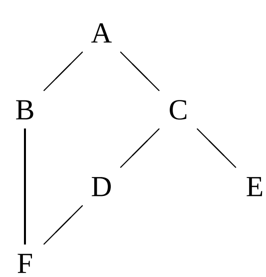
Note that $\langle\cdot\rangle_A$ requires every type variabe α_i to be a fresh type variable, unique in its context. This ensures that we do not inadvertently refer to any type variable bound in the context where we are introducing the abstractly encoded type.

As an example, consider the powerset lattice of $\{1, 2, 3, 4\}$, which encodes into a four-tuple. We can verify, for example, that the concrete encoding for $\{2\}$, namely (unit z $\times$ unit $\times$ unit z $\times$ unit z), unifies with the abstract encoding for $\{1, 2\}$, namely (α_1 $\times$ α_2 $\times$ α_3 z $\times$ α_4 z). On the other hand, the concrete encoding of $\{1, 2\}$ does not unify with the abstract encoding of $\{2, 3\}$.

3.3 Embeddings

The main reason we introduced powerset lattices is the fact that any finite hierarchy can be embedded in the powerset lattice of a set S. It is a simple matter, given a hierarchy H' embedded in a hierarchy H, to derive an encoding for H' given an encoding for H. Let $inj(\cdot)$ be the injection from H' to H witnessing the embedding and let $\langle\cdot\rangle_{C_H}$ and $\langle\cdot\rangle_{A_H}$ be the encodings for the hierarchy H. Deriving an encoding for H' simply involves defining $\langle\sigma\rangle_{C_{H'}} = \langle inj(\sigma)\rangle_{C_H}$ and $\langle\sigma\rangle_{A_{H'}} = \langle inj(\sigma)\rangle_{A_H}$. It is straightforward to verify that if $\langle\cdot\rangle_{C_H}$ and $\langle\cdot\rangle_{A_H}$ are respectful encodings, so are $\langle\cdot\rangle_{C_{H'}}$ and $\langle\cdot\rangle_{A_{H'}}$. By the result above, this allows us to derive an encoding for an arbitrary finite hierarchy.

To give an example of embedding, consider the following subtyping hierarchy to be encoded:



Notice that this lattice can be embedded into the powerset lattice of $\{1, 2, 3, 4\}$, via the injection function sending A to $\{1, 2, 3, 4\}$, B to $\{1, 2, 3\}$, C to $\{2, 3, 4\}$, D to $\{2, 3\}$, E to $\{3, 4\}$, and F to $\{2\}$.

3.4 Other encodings

As we noted, the key issue in modeling a subtyping relation over an abstract type is to be able to encode the subtyping hierarchy. We have presented recipes for obtaining respectful encodings, depending on the characteristics of the hierarchy at hand. It should be clear that there are more general hierarchies than the ones presented here that can still be encoded, although the encodings quickly become complicated.

Not only do we have flexibility in designing the encoding of the subtyping hierarchy under consideration, we can also derive encodings for general hierarchies with the understanding that only a restricted class of elements will ever be accessed. As an example, let us examine an encoding that generalizes the finite powerset lattice encoding to the (countably) infinite case. This encoding is different from the other encodings, in the sense that we can in fact encode only the finite subsets of a countably infinite set. On the other hand, this case is interesting enough to warrant a discussion.

Technically, the encoding is in the spirit of the finite powerset encoding. Let S be a countably infinite set, and assume an ordering $s_1, s_2, \ldots$ of the elements of S. As in the finite case, we define a datatype

```
datatype 'a z = Z
```

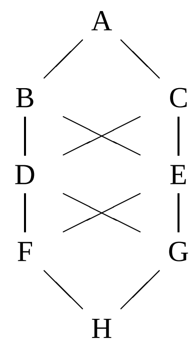
The encoding is given for *finite* subsets of S by the following pair of encodings:

$$\begin{aligned}
\langle X\rangle_C &= (t_1 \times (t_2 \times (t_3 \times \ldots \times (t_n \times \alpha)\ldots))) \\
&\quad \text{where } t_i = \begin{cases} \texttt{unit} & \text{if } s_i \in X \\ \texttt{unit z} & \text{otherwise} \end{cases} \\
&\quad\quad \text{and } n \text{ is the least index such that } X \subseteq \{s_1, \ldots, s_n\} \\
\langle X\rangle_A &= (t_1 \times (t_2 \times (t_3 \times \ldots \times (t_n \times \texttt{unit}), \ldots))) \\
&\quad \text{where } t_i = \begin{cases} \alpha_i & \text{if } s_i \in X \\ \alpha_i\ \texttt{z} & \text{otherwise} \end{cases} \\
&\quad\quad \text{and } n \text{ is the least index such that } X \subseteq \{s_1, \ldots, s_n\}
\end{aligned}$$

(As usual, with the restriction that the type variables $\alpha, \alpha_1, \ldots, \alpha_n$ are fresh.) One can verify that this is indeed a respectful encoding of the finite elements of the infinite lattice.

Note that this use of a free type variable in the encoding to basically be "polymorphic in the rest of the encoded value" is strongly reminiscent of the notion of a *row variable*, as originally used by Wand to type extensible records [Wand 1987], and further developped by Rémy [1989]. The technique was used by Burton to encode extensible records directly in a polymorphic type systems [Burton 1990]. Recently, the same technique was used to represents sets of effects in type-and-effect systems [Pessaux and Leroy 1999].

We have not focussed on the complexity or space-efficiency of the encodings. We have emphasized simplicity and uniformity of the encodings, at the expense of succinctness. For instance, deriving an encoding for a finite hierarchy by embedding it in a powerset lattice can lead to large encodings even when simpler encodings exist. Consider the following subtyping hierarchy:



The smallest powerset lattice in which this hierarchy can be embedded is the powerset lattice of a 6-element set, and therefore the encoding will require 6-tuples. On the other hand, it is not hard to verify that the following encoding respect the subtyping induced by this hierarchy. First assume a datatype declaration:

```
datatype 'a l = L
```

(Again, the constructor name is irrelevant.) Consider the following encoding:

$$
\begin{aligned}
\langle A\rangle_C &= (\texttt{unit} \times \texttt{unit}) \\
\langle B\rangle_C &= (\texttt{unit l} \times \texttt{unit}) \\
\langle C\rangle_C &= (\texttt{unit} \times \texttt{unit l}) \\
\langle D\rangle_C &= (\texttt{(unit l) l} \times \texttt{unit l}) \\
\langle E\rangle_C &= (\texttt{unit l} \times \texttt{(unit l) l}) \\
\langle F\rangle_C &= (\texttt{((unit l) l) l} \times \texttt{(unit l) l}) \\
\langle G\rangle_C &= (\texttt{(unit l) l} \times \texttt{((unit l) l) l}) \\
\langle H\rangle_C &= (\texttt{((unit l) l) l} \times \texttt{((unit l) l) l})
\end{aligned}
$$

(The abstract encoding is obtained by replacing every `unit` by a type variable α, taken fresh, as usual.)

It is possible to generate encodings for finite hierarchies that are in general more efficient than the encodings derived from the powerset lattice embeddings. One such encoding, described in Appendix A, uses a tuple approach just like the powerset lattice encoding. This encoding yields tuples whose size correspond to the width of the subtyping hierarchy being encoded,[2] rather than the typically larger size of the smallest set in whose powerset lattice the hierarchy can be embedded. (The efficient encoding for the previous subtyping hierarchy is an instance of such a width encoding.)

3.5 Encodings and extensibility

One aspect of encodings we have not yet discussed is that of extensibility. Roughly speaking, extensibility refers to the possibility of adding new elements to the subtyping hierarchy after a program has already been written. One would like to avoid having to rewrite the whole program taking the new subtyping hierarchy into account. This is especially important in the design of

[2]The *width* of a finite hierarchy is the size of the largest set of mutually incomparable elements.

libraries, where the user may need to extend the kind of data that the library handles, without changing the provided interface. In this section, we examine the extensibility of the encodings in Section 3. We then show how to capture one type of extensibility, restricted but still useful, through any encoding.

Looking at the encodings of Section 3, it should be clear that the only immediately extensible encodings are the tree encodings in Section 3.1. In such a case, adding a new subtype S to a given type T in the tree simply requires the definition of a new datatype:

$$\texttt{datatype 'a } \langle S\rangle_N \texttt{ = C\_}\langle S\rangle_N$$

We assume a naming function $\langle\cdot\rangle_N$ extended to include S. For the sake of presentation, we assume $\langle S\rangle_N = \texttt{LS}$. One can check that the abstract and concrete enc odings of the original elements of the hierarchy are not changed by the extension—since the encoding relies on the path to the elements. The concrete and abstract encodings of the new subtype S is just the path to S, as expected.

The powerset lattice encodings and their embeddings are not so clearly extensible. Indeed, in general, any encoding of a subtyping hierarchy that contains "join elements" (that is, a type which is a subtype of at least two otherwise unrelated types, related to *multiple inheritance* in object-oriented programming) will not be extensible in an arbitrary way. However, it turns out that as long as all the extensions are done with respect to a single "parent" type, it is possible to extend any subtyping hierarchy in a way that does not invalidate the previous encoding. Let us take as a starting point the finite powerset encoding of Section 3.2. Observe that in the lattice encodings, the encoding of a type T corresponding to subset $\{t_{i_1},\ldots,t_{i_n}\}$ contains a `unit` in the tuple position corresponding to t_{i_1} to t_{i_n}, and `unit out` in other positions. Suppose we wish to derive a new subtype S of T (not in the original lattice); we simply encode it as we would T, but creating a new type, say:

```
datatype 'a LS = C_LS
```

as in the case of tree encodings, and replace every `unit` in the encoding of T by `unit LS`. For the abstract encoding of S, we replace every `unit` in the concrete encoding by a (fresh) type variable. One can verify that indeed the resulting encoding is respectful of the updated subtyping hierarchy.

In general, we can extend a lattice encoding by "grafting" another lattice to any node. Here is a general recipe to achieve this. Let L be a powerset lattice over a set S of cardinality n, and let σ be an element of L we want to extend by another sublattice L_σ, that is, everything in L_σ is a subtype of σ. (Recall that σ is in fact a subset of S.) Assume that L is encoded via a lattice embedding encoding $\langle\cdot\rangle_C, \langle\cdot\rangle_A$, and that L_σ is encoded via some encoding $\langle\cdot\rangle^\sigma_C, \langle\cdot\rangle^\sigma_A$. We can extend the encoding for L over the elements of L_σ:

$$\begin{aligned}
\langle\tau\rangle_C &= t_1\times\ldots\times t_n \quad \text{where } t_i = \begin{cases}\langle\tau\rangle^\sigma_C & \text{if } s_i\in\sigma\\ \texttt{unit z} & \text{otherwise}\end{cases}\\
\langle\tau\rangle_A &= t_1\times\ldots\times t_n \quad \text{where } t_i = \begin{cases}\langle\tau\rangle^\sigma_A & \text{if } s_i\in\sigma\\ \alpha_i\ \texttt{z} & \text{otherwise}\end{cases}
\end{aligned}$$

(As usual, each α_i in $\langle\cdot\rangle_A$ is fresh, including the type variables in $\langle\tau\rangle^\sigma_A$.) Again, such an encoding is easily seen as being respectful of the extended subtyping hierarchy. The above scheme generalizes in the straightforward way to encodings via lattice embeddings, and to the countable lattice encoding of Section 3.4.

The interesting thing to notice about the above development is that although extensions are restricted to a single type (i.e., we can only subtype one given type), the extension can itself be an arbitrary lattice. It does not seem possible to describe a general extensible encoding that supports subtyping two different types at the same time (multiple inheritance). In other words, to adopt an object-oriented perspective, we cannot multiply-inherit from another type, but we can single-inherit into an arbitrary lattice, which can use multiple inheritance locally.

4 Towards bounded polymorphism

As mentioned in Section 3, the handling of type variables is somewhat delicate. If we allow common type variables to be used across abstract encodings, then we can capture a form of *bounded polymorphism* as in $F_{<:}$ [Cardelli, Martini, Mitchell, and Scedrov 1994]. Bounded polymorphism *à la* $F_{<:}$ is a typing discipline which extends both parametric polymorphism and subtyping. From parametric polymorphism, it borrows type variables and universal quantification; from subtyping, it allows one to set bounds on quantified type variables. For example, one can guarantee that the argument and return types of a function are the same and a subtype of σ, as in $\forall\alpha \leq \sigma.\alpha \rightarrow \alpha$. Similarly, one can guarantee that two arguments have the same type that is a subtype of σ, as in $\forall\alpha \leq \sigma.(\alpha \times \alpha) \rightarrow \sigma$. Notice that neither function can be written in a language that supports only subtyping.

Returning to the example from the introduction, consider adding natural numbers as a subtype of integers, so that *nat* is a subtype of *int*. Using bounded polymorphism, we can assign to `double` the reasonable type $\forall\alpha \leq \texttt{int}.\alpha \rightarrow \alpha$. However, bounded polymorphism has its limitations. One reasonable type for a `plus` operation is $\forall\alpha \leq \texttt{int}.\alpha \times \alpha \rightarrow \alpha$ where the same kind of atom is required for both arguments. In order to add an integer and a natural number we need a function `toInt` (operationally, an identity function) to coerce the type of the natural number to that of an integer.

We can adapt our "recipe" from Section 2 to types of the form $\forall\beta \leq \sigma_1.(\beta \times \sigma_2) \rightarrow \beta$. Let the "safe" interface use types of the form $\alpha\ \tau$. Since β stands for a subtype of σ_1, we let $\phi_\beta = \langle\sigma_1\rangle_A$, the abstract encoding of the bound. We then translate the type as we did in Section 2, but replace occurrences of the type variable β by ϕ_β instead of applying $\langle\cdot\rangle_A$ repeatedly, thereby sharing the type variables introduced by $\langle\sigma_1\rangle_A$. Hence, we get the type $\phi_\beta\ \tau \times \langle\sigma_2\rangle_A\ \tau \rightarrow \phi_\beta\ \tau$. In fact, we can further simplify the process by noting that we can "pull out" all the subtyping into bounded polymorphism. If a function expects an argument of any implicit subtype of σ, we can introduce a fresh type variable for that argument and bound it by σ. For example, the type above can be rewritten as: $\forall\beta \leq \sigma_1, \gamma \leq \sigma_2.(\beta \times \gamma) \rightarrow \beta$.

Unfortunately, this technique does not generalize to full $F_{<:}$. For example, we cannot encode bounded polymorphism where the bound on a type variable uses a type variable, such as a function `f` with type $\forall\alpha \leq \sigma, \beta \leq \alpha.\alpha \times \beta \rightarrow \alpha$. Encoding this type as $\phi_\alpha\ \tau \times \phi_\beta\ \tau \rightarrow \phi_\alpha\ \tau$ where $\phi_\alpha = \langle\sigma\rangle_A$ and $\phi_\beta = \langle\alpha\rangle_A$ fails, because we have no definition of $\langle\alpha\rangle_A$. Essentially, we need a different encoding of β for each instantiation of α at each application of `f`, something that cannot be accommodated by a single encoding of the type at the definition of `f`.

Likewise, we cannot encode first-class polymorphism, such as a function `g` with type $\forall\alpha \leq \sigma_1.\alpha \rightarrow (\forall\beta \leq \sigma_2.\beta \rightarrow \beta)$. Applying the technique yields a type $\phi_\alpha\ \tau \rightarrow \phi_\beta\ \tau \rightarrow \phi_\beta\ \tau$ where ϕ_α and ϕ_β contain free type variables. A Hindley-Milner style type system requires quantification over these variables in prenex position, which doesn't match the intuition of the original type. In fact, because we are translating into a language with prenex polymorphism, we can only capture bounded polymorphism that is itself in prenex form.

In other words, we cannot account for the general subsumption rule found in $F_{<:}$. Instead, we require all subtyping to occur at type application. This is the real motivation for the simplification above which "pulls out" all subtyping into bounded polymorphism. By introducing type variables for each argument, we move the resolution of the subtyping to the point of type application (when we instantiate the type variables). The following example may illustrate this point. In $F_{<:}$ with first-class polymorphism, we can write a function `app` with type $(\forall \alpha \le \sigma_1 . \alpha \to \sigma_2) \to \sigma_2 \times \sigma_2$ that applies a function to two values, `v1` of type σ_1 and `v2` of type $\sigma_2 \le \sigma_1$. This definition of `app` type-checks when we apply the argument function to σ_1 and then to `v1` and we apply the argument function to σ_2 (using subsumption at type application) and then to `v2`. But, as we argued above, we cannot encode first-class polymorphism. An alternative version of `app` can be written in $F_{<:}$ with type $(\sigma_1 \to \sigma_2) \to \sigma_2 \times \sigma_2$. This definition of `app` type-checks when we apply the argument function to `v1` and we apply the argument function to `v2` (using subsumption on the argument function to give it the type $\sigma_2 \to \sigma_2$). Yet, we cannot give any reasonable encoding of `app` into SML, because it would require applying the argument function to the encoding of `v1`, with type $\langle \sigma_1 \rangle_C \ \tau$, and to the encoding of `v2`, with type $\langle \sigma_2 \rangle_C \ \tau$; that is, it would require applying an argument function at two different types. As hinted above, this is a consequence of the lack of first-class polymorphism in the SML type system; the argument function cannot be polymorphic.

These two restrictions impose one final restriction on the kind of subtyping we can encode. Consider a higher-order function `h` with type $\alpha \le (\sigma_1 \to \sigma_2) . \alpha \to \sigma_2$. What are the possible encodings of the bound $\sigma_1 \to \sigma_2$ that allow subtyping? Clearly encoding the bound as $\langle \sigma_1 \rangle_C \ \tau \to \langle \sigma_2 \rangle_C \ \tau$ does not allow any subtyping. Encoding the bound as $\langle \sigma_1 \rangle_A \ \tau \to \langle \sigma_2 \rangle_A \ \tau$ or $\langle \sigma_1 \rangle_A \ \tau \to \langle \sigma_2 \rangle_C \ \tau$ leads to an unsound system. (Consider applying the argument function to a value of type $\sigma_0 \ge \sigma_1$, which would type-check in the encoding, because $\langle \sigma_0 \rangle_C$ unifies with $\langle \sigma_1 \rangle_A$ by the definition of a respectful encoding.) However, we can soundly encode the bound as $\langle \sigma_1 \rangle_C \ \tau \to \langle \sigma_2 \rangle_A \ \tau$. This corresponds to a subtyping rule on functional types that asserts $\tau_1 \to \tau_2 \le \tau_1 \to \tau_2'$ iff $\tau_2 \le \tau_2'$.

Despite these restrictions, the phantom types technique is still a viable method for encoding subtyping in a language like SML. All of the examples of phantom types found in the literature satisfy these restrictions. In practice, one rarely needs first-class polymorphism or complicated dependencies between the subtypes of function arguments, particularly when implementing a safe interface to existing library functions.

5 A formalization

There are subtle issues regarding the kind of subtyping that can be captured using phantom types. In this section, we clarify the picture by exhibiting a typed calculus with a suitable notion of subtyping that can be faithfully translated into a language such as SML, via a phantom types encoding. The idea is simple: to see if an interface can be implemented using phantom types, first express the interface in this calculus in such a way that the program type-checks. If it is possible to do so, our results show that a translation using phantom types exists. The target of the translation is a calculus embodying the essence of SML, essentially the calculus of Damas and Milner [1982], a predicative polymorphic λ-calculus.

5.1 The source calculus $\lambda^{\mathrm{DM}}_{<:}$

Our source calculus, $\lambda^{\mathrm{DM}}_{<:}$, is a variant of the Damas-Milner calculus with a very restricted notion of subtyping, and allowing multiple types for constants. We assume a partially ordered set $(T, \leq)$ of basic types.

Types of $\lambda^{\mathrm{DM}}_{<:}$:

$\tau ::=$	Monotypes
$\quad t$	$\quad$ Basic type $(t \in T)$
$\quad \alpha$	$\quad$ Type variable
$\quad \tau_1 \to \tau_2$	$\quad$ Function type
$\sigma ::=$	Prenex quantified type scheme
$\quad \forall \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n . \tau$	

We make a syntactic restriction that precludes the use of type variables in the bounds of quantified type variables.

An important aspect of our calculus, at least for our purposes, is the constants that we allow. We distinguish between two types of constants: basic constants and primitive operations. Basic constants, taken from a set C_b, are constants representing values of basic types $t \in T$. We suppose a function $\pi_b : C_b \to T$ assigning a basic type to every basic constant. The primitive operations, taken from a set C_p, are operations acting on constants and returning constants.[3] Rather than giving primitive operations polymorphic types, we assume that the operations can have multiple types, which encode the allowed subtyping. The primitive operation `double` in our example would get the types `int_value` $\to$ `int_value` and `nat_value` $\to$ `nat_value`. We suppose a function π_p assigning to every constant $c \in C_p$ a set of types $\pi_p(c)$, each type a functional type of the form $t \to t'$ (for $t, t' \in T$).

Our expression language is a typical polymorphic lambda calculus expression language.

Expression syntax of $\lambda^{\mathrm{DM}}_{<:}$:

$e ::=$	Monomorphic expressions
$\quad c$	$\quad$ Constant $(c \in C_b \cup C_p)$
$\quad \lambda x : \tau . e$	$\quad$ Functional abstraction
$\quad e_1\ e_2$	$\quad$ Function application
$\quad x$	$\quad$ Variable
$\quad p\ [\tau_1, \ldots, \tau_n]$	$\quad$ Type application
$\quad$ **let** $x = p$ **in** e	$\quad$ Local binding
$p ::=$	Polymorphic expressions
$\quad x$	$\quad$ Variable
$\quad \Lambda \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n . e$	$\quad$ Type abstraction

The operational semantics are given using a standard contextual reduction semantics, written $e_1 \longrightarrow_{<:} e_2$. The details can be found in Appendix B, but we note the most important reduction

[3]For simplicity, we will not deal with higher-order functions here—they would simply complicate the formalism without bringing any new insight. Likewise, allowing primitive operations to act on and return tuples of values is a simple extension of the formalism presented here.

rule, involving constants:

$$c_1\ c_2 \longrightarrow_{<:} c_3 \text{ iff } \delta(c_1, c_2) = c_3$$

where $\delta : C_p \times C_b \rightharpoonup C_p$ is a partial function defining the result of applying a primitive operation to a basic constant.

As previously noted, we only allow primitive operations to be monotyped. However, we can easily use the fact that they can take on many types to write polymorphic wrappers. Returning to the `double` example, we can write a polymorphic wrapper $\Lambda\alpha <: \texttt{int\_value}.\lambda x : \alpha.\texttt{double}\ x$ to capture the expected behavior. We will see shortly that this function is well-typed.

The typing rules for $\lambda^{\mathrm{DM}}_{<:}$ are the standard Damas-Milner typing rules, modified to account for subtyping. The full set of rules is given in Appendix B. Subtyping is given by a judgment $\Delta \vdash_{<:} \tau_1 <: \tau_2$, and is derived from the subtyping on the basic types. The interesting rules are:

$$\frac{t_1 \leq t_2}{\Delta \vdash_{<:} t_1 <: t_2} \qquad \frac{\Delta \vdash_{<:} \tau_2 <: \tau_2'}{\Delta \vdash_{<:} \tau_1 \rightarrow \tau_2 <: \tau_1 \rightarrow \tau_2'}$$

Notice that subtyping at higher types only involves the result type. The typing rules are given by judgments $\Delta; \Gamma \vdash_{<:} e : \tau$ for monotypes and $\Delta; \Gamma \vdash_{<:} p : \sigma$ for type schemes. The rule for primitive operations is interesting:

$$\frac{(\tau' \rightarrow \tau)\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\} \in \pi_p(c) \quad \forall \tau_1' <: \tau_1, \ldots, \tau_n' <: \tau_n}{\Delta, \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n; \Gamma \vdash_{<:} c : \tau} \quad \left(\begin{array}{c} c \in C_p, \\ FV(\tau') = \langle \alpha_1, \ldots, \alpha_n \rangle \end{array}\right)$$

The syntactic restriction on type variable bounds ensures that each τ_i has no type variables, so each $\tau_i' <: \tau_i$ is well-defined. The rule captures the notion that any subtyping on a primitive operation through the use of bounded polymorphism is in fact realized by the "many types" interpretation of the operation.

Subtyping occurs at type application:

$$\frac{\Delta; \Gamma \vdash_{<:} p : \forall \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau \quad \Delta \vdash_{<:} \tau_1' <: \tau_1 \quad \cdots \quad \Delta \vdash_{<:} \tau_n' <: \tau_n}{\Delta; \Gamma \vdash_{<:} p\ [\tau_1', \ldots, \tau_n'] : \tau\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\}}$$

As discussed in the previous section, there is no subsumption in the system: subtyping must be witnessed by type application. Hence, there is a difference between the type $t_1 \rightarrow t_2$ (where $t_1, t_2 \in T$) and $\forall \alpha <: t_1.\alpha \rightarrow t_2$; namely, the former does not allow any subtyping. The restrictions of Section 4 are formalized by prenex quantification and the syntactic restriction on type variable bounds.

Clearly, type soundness of the above system depends on the definition of δ over the constants. We say that π_p is sound with respect to δ if for all $c_1 \in C_p$ and $c_2 \in C_b$, we have $\vdash_{<:} c_1\ c_2 : \tau$ implies that $\delta(c_1, c_2)$ is defined and $\pi_b(\delta(c_1, c_2)) = \tau$. This definition ensures that any application of a primitive operation c_1 to a basic constant c_2 yields exactly one value $\delta(c_1, c_2)$ at exactly one type $\pi_b(\delta(c_1, c_2)) = \tau$. This leads to the following conditional type soundness result for $\lambda^{\mathrm{DM}}_{<:}$:

Theorem 5.1 *If π_p is sound with respect to δ, $\vdash_{<:} e : \tau$, and $e \longrightarrow_{<:} e'$, then $\vdash_{<:} e' : \tau$ and either e' is a value or there exists e'' such that $e' \longrightarrow_{<:} e''$.*

Proof: See Appendix B. ∎

5.2 The target calculus $\lambda_{\mathsf{T}}^{\mathrm{DM}}$

Our target calculus, $\lambda_{\mathsf{T}}^{\mathrm{DM}}$, is meant to capture the appropriate aspects of SML that are relevant for the phantom types encoding of subtyping. Essentially, it is the Damas-Milner calculus [Damas and Milner 1982] extended with a single type constructor T.

Types of $\lambda_{\mathsf{T}}^{\mathrm{DM}}$:

$\tau ::=$	Monotypes
α	Type variable
$\tau_1 \to \tau_2$	Function type
$\mathsf{T}\ \tau$	Type constructor T
1	Unit type
$\tau_1 \times \tau_2$	Product type
$\sigma ::=$	Prenex quantified type scheme
$\forall \alpha_1, \ldots, \alpha_n.\tau$	

Expression syntax of $\lambda_{\mathsf{T}}^{\mathrm{DM}}$:

$e ::=$	Monomorphic expressions
c	Constant $(c \in C_b \cup C_p)$
$\lambda x\!:\!\tau.e$	Functional abstraction
$e_1\ e_2$	Function application
x	Variable
$p\ [\tau_1, \ldots, \tau_n]$	Type application
let $x = p$ **in** e	Local binding
$p ::=$	Polymorphic expressions
x	Variable
$\Lambda\alpha_1 \ldots, \alpha_n.e$	Type abstraction

The operational semantics (via a reduction relation $\longrightarrow_{\mathsf{T}}$) and most typing rules (via a judgment $\Delta; \Gamma \vdash_{\mathsf{T}} e : \tau$) are standard. The calculus is fully described in Appendix C. As before, we assume that we have constants C_b and C_p and a function δ providing semantics for primitive applications. Likewise, we assume that π_b and π_p provide types for constants, with similar restrictions: $\pi_b(c)$ yields a closed type of the form $\mathsf{T}\ \tau$, while $\pi_p(c)$ yields a set of closed types of the form $(\mathsf{T}\ \tau_1) \to (\mathsf{T}\ \tau_2)$. The typing rule for primitive operations in $\lambda_{\mathsf{T}}^{\mathrm{DM}}$ is similar to the corresponding rule in $\lambda_{<:}^{\mathrm{DM}}$. Given two types τ and τ' in $\lambda_{\mathsf{T}}^{\mathrm{DM}}$, we define their unification $\mathit{unify}(\tau, \tau')$ to be a sequence of bindings $\langle(\alpha_1, \tau_1), (\alpha_2, \tau_2), \ldots\rangle$ in depth-first, left-to-right order of appearance of $\alpha_1, \ldots, \alpha_n$ in τ, or $\emptyset$ if τ' is not a substitution instance of τ. Given a type τ in $\lambda_{\mathsf{T}}^{\mathrm{DM}}$, we define $FV(\tau)$ to be the sequence of free type variables appearing in τ, in depth-first, left-to-right order.

$$\frac{\begin{array}{c}\forall \tau' \in \pi_b(C_b) \text{ with } \mathit{unify}(\tau_1, \tau') = \langle(\alpha_1, \tau_1'), \ldots, (\alpha_n, \tau_n'), \ldots\rangle \\ (\tau_1 \to \tau_2)\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\} \in \pi_p(c)\end{array}}{\Delta, \alpha_1, \ldots, \alpha_n; \Gamma \vdash_{\mathsf{T}} c : \tau_1 \to \tau_2} \quad \left(\begin{array}{c} c \in C_p, \\ FV(\tau_1) = \langle\alpha_1, \ldots, \alpha_n\rangle \end{array}\right)$$

Again, this rule captures our notion of "subtyping through unification" by ensuring that the operation is defined at every basic type that unifies with its argument type. Our notion of soundness of π_p with respect to δ carries over and we can again establish a conditional type soundness result:

Theorem 5.2 *If π_p is sound with respect to δ, $\vdash_{\mathsf{T}} e : \tau$, and $e \longrightarrow_{\mathsf{T}} e'$, then $\vdash_{\mathsf{T}} e' : \tau$ and either e' is a value or there exists e'' such that $e' \longrightarrow_{\mathsf{T}} e''$.*

Proof: See Appendix C. ∎

Note that the types $\mathsf{T}\ \tau$, 1, and $\tau_1 \times \tau_2$ have no corresponding introduction and elimination expressions. We include these types for the exclusive purpose of constructing the phantom types used by the encodings. We could add other types to allow more encodings, but these suffice for the lattice encodings of Section 3.

5.3 The translation

Thus far, we have a calculus $\lambda_{<:}^{\mathrm{DM}}$ embodying the notion of subtyping that interests us and a calculus $\lambda_{\mathsf{T}}^{\mathrm{DM}}$ capturing the essence of the SML type system. We now establish a translation from the first calculus into the second using phantom types to encode the subtyping, showing that we can indeed capture that particular notion of subtyping in SML. Moreover, we show that the translation preserves the soundness of the types assigned to constants, thereby guaranteeing that if the original system was sound, the system obtained by translation is sound as well.

We first describe how to translate types in $\lambda_{<:}^{\mathrm{DM}}$. Since subtyping is only witnessed at type abstraction, the type translation realizes the subtyping using the phantom types encoding of abstract and concrete subtypes. The translation is parameterized by an environment ρ associating every (free) type variable with a type in $\lambda_{\mathsf{T}}^{\mathrm{DM}}$ representing the abstract encoding of the bound.

Types translation:

$$\mathcal{T}[\![\alpha]\!]\rho \triangleq \rho(\alpha)$$
$$\mathcal{T}[\![t]\!]\rho \triangleq \mathsf{T}\ \langle t\rangle_C$$
$$\mathcal{T}[\![\tau_1 \to \tau_2]\!]\rho \triangleq \mathcal{T}[\![\tau_1]\!]\rho \to \mathcal{T}[\![\tau_2]\!]\rho$$
$$\mathcal{T}[\![\forall\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau]\!]\rho \triangleq \forall\alpha_{11}, \ldots, \alpha_{1k_1}, \ldots, \alpha_{n1}, \ldots, \alpha_{nk_n}.\mathcal{T}[\![\tau]\!]\rho[\alpha_i \mapsto \tau_i^A]$$
$$\text{where } \tau_i^A = \mathcal{A}[\![\tau_i]\!] \text{ and } FV(\tau_i^A) = \langle \alpha_{i1}, \ldots, \alpha_{ik_i}\rangle$$

If ρ is empty, we will simple write $\mathcal{T}[\![\tau]\!]$. To compute the abstract and concrete encodings of a type, we define the functions $\mathcal{A}$ and $\mathcal{C}$.

Abstract and concrete encodings:

$$\mathcal{A}[\![t]\!] \triangleq \mathsf{T}\ \langle t\rangle_A$$
$$\mathcal{A}[\![\tau_1 \to \tau_2]\!] \triangleq \mathcal{C}[\![\tau_1]\!] \to \mathcal{A}[\![\tau_2]\!]$$
$$\mathcal{C}[\![t]\!] \triangleq \mathsf{T}\ \langle t\rangle_C$$
$$\mathcal{C}[\![\tau_1 \to \tau_2]\!] \triangleq \mathcal{C}[\![\tau_1]\!] \to \mathcal{C}[\![\tau_2]\!]$$

Note that the syntactic restriction on type variable bounds ensures that $\mathcal{A}$ and $\mathcal{C}$ are always well-defined, as they will never be applied to type variables. Furthermore, observe that the above translation depends on the fact that the type encodings $\langle t\rangle_C$ and $\langle t\rangle_A$ are expressible in the $\lambda_{\mathsf{T}}^{\mathrm{DM}}$ type system using T, 1, and $\times$.

We extend the type transformation $\mathcal{T}$ to type contexts Γ in the obvious way:

Type contexts translation:

$$\mathcal{T}[\![x_1 : \tau_1, \ldots, x_n : \tau_n]\!]\rho \triangleq x_1 : \mathcal{T}[\![\tau_1]\!]\rho, \ldots, x_n : \mathcal{T}[\![\tau_n]\!]\rho$$

Finally, if we take the basic constants and the primitive operations in $\lambda^{\mathrm{DM}}_{<:}$ and assume that π_p is sound with respect to δ, then the translation can be used to assign types to the constants and operations such that they are sound in the target calculus. We first extend the definition of $\mathcal{T}$ to π_b and π_p in the obvious way:

Interpretations translation:

$$\mathcal{T}[\![\pi_b]\!] \triangleq \pi'_b \quad \text{where } \pi'_b(c) = \mathcal{T}[\![\pi_b(c)]\!]$$
$$\mathcal{T}[\![\pi_p]\!] \triangleq \pi'_p \quad \text{where } \pi'_p(c) = \{\mathcal{T}[\![\tau]\!] \mid \tau \in \pi_p(c)\}$$

We can further show that the translated types do not allow us to "misuse" the constants in $\lambda^{\mathrm{DM}}_{\mathsf{T}}$:

Theorem 5.3 *If π_p is sound with respect to δ in $\lambda^{\mathrm{DM}}_{<:}$, then $\mathcal{T}[\![\pi_p]\!]$ is sound with respect to δ in $\lambda^{\mathrm{DM}}_{\mathsf{T}}$.*

Proof: See Appendix D. ∎ We therefore take $\mathcal{T}[\![\pi_b]\!]$ and $\mathcal{T}[\![\pi_p]\!]$ to be the interpretations in target calculus $\lambda^{\mathrm{DM}}_{\mathsf{T}}$.

We can now define the translation of expressions via a translation of typing derivations, $\mathcal{E}$, taking care to respect the types given by the above type translation. We note that the translation below only works if the concrete encodings being used do not contain free type variables. Again, the translation is parameterized by an environment ρ, as in the type translation.

Expressions translation:

$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} x : \tau]\!]\rho \triangleq x$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} c : \tau]\!]\rho \triangleq c$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} \lambda x{:}\tau'.e : \tau]\!]\rho \triangleq \lambda x{:}\mathcal{T}[\![\tau']\!]\rho.\mathcal{E}[\![e]\!]\rho$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} e_1\ e_2 : \tau]\!]\rho \triangleq (\mathcal{E}[\![e_1]\!]\rho)\ \mathcal{E}[\![e_2]\!]\rho$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} \textbf{let } x = p \textbf{ in } e : \tau]\!]\rho \triangleq \textbf{let } x = \mathcal{E}[\![p]\!]\rho \textbf{ in } \mathcal{E}[\![e]\!]\rho$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} p\ [\tau_1, \ldots, \tau_n] : \tau]\!]\rho \triangleq$$
$$(\mathcal{E}[\![p]\!]\rho)\ [\tau_{11}, \cdots, \tau_{1k_1}, \cdots, \tau_{n1}, \cdots, \tau_{nk_n}]$$
$$\text{where } \mathcal{B}[\![p]\!]\Gamma = (\alpha_1, \tau_1^B), \ldots, (\alpha_n, \tau_n^B)\rangle \text{ and } \tau_i^A = \mathcal{A}[\![\tau_i^B]\!]$$
$$\text{and } FV(\tau_i^B) = \langle \alpha_{i1}, \ldots, \alpha_{ik_i}\rangle \text{ and } \tau_i^T = \mathcal{T}[\![\tau_i]\!]\rho$$
$$\text{and } \mathit{unify}(\tau_i^A, \tau_i^T) = \langle(\alpha_{i1}, \tau_{i1}), \ldots, (\alpha_{ik_i}, \tau_{ik_i}), \ldots\rangle$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} x : \sigma]\!]\rho \triangleq x$$
$$\mathcal{E}[\![\Delta;\Gamma \vdash_{<:} \Lambda\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.e : \sigma]\!]\rho \triangleq$$
$$\Lambda\alpha_{11}, \ldots, \alpha_{1k_1}, \ldots, \alpha_{n1}, \ldots, \alpha_{nk_n}.\mathcal{E}[\![e]\!]\rho[\alpha_i \mapsto \tau_i^A]$$
$$\text{where } \tau_i^A = \mathcal{A}[\![\tau_i]\!] \text{ and } FV(\tau_i^A) = \langle \alpha_{i1}, \ldots, \alpha_{ik_i}\rangle$$

Again, if ρ is empty, we simply write $\mathcal{E}[\![e]\!]$. The function $\mathcal{B}$ returns the bounds of a type abstraction, using the environment Γ to resolve variables.

Bounds of a type abstraction:

$\mathcal{B}[\![x]\!]\Gamma \triangleq \langle(\alpha_1,\tau_1),\ldots,(\alpha_n,\tau_n)\rangle$ where $\Gamma(x) = \forall\alpha_1 <: \tau_1,\ldots,\alpha_n <: \tau_n.\tau$
$\mathcal{B}[\![\Lambda\alpha_1 <: \tau_1,\ldots,\alpha_n <: \tau_n.e]\!]\Gamma \triangleq \langle(\alpha_1,\tau_1),\ldots,(\alpha_n,\tau_n)\rangle$

We use $\mathcal{B}$ and $unify$ to perform unification "by hand." In most programming languages, type inference performs this automatically.

We can verify that this translation is type-preserving:

Theorem 5.4 *If* $\vdash_{<:} e : \tau$*, then* $\vdash_{\mathsf{T}} \mathcal{E}[\![\vdash_{<:} e : \tau]\!] : \mathcal{T}[\![\tau]\!]$.

Proof: See Appendix D. ∎

Theorem 5.4 is interesting in that it shows that the translation, in a sense, captures the right notion of subtyping, particularly when designing an interface. Given a set of constants making up the interface, suppose we can assign types to those constants in $\lambda^{\mathrm{DM}}_{<:}$ in a way that gives the desired subtyping; that is, we can write type correct expressions of the form $\Lambda\alpha <: t.\lambda x : \alpha.c\ x$ with type $\forall\alpha <: t.\alpha \rightarrow \tau$. In other words, the typing π_p is sound with respect to the semantics of δ. By Theorem 5.1, this means that $\lambda^{\mathrm{DM}}_{<:}$ with these constants is sound and we can safely use these constants in $\lambda^{\mathrm{DM}}_{<:}$. In particular, we can write the program:

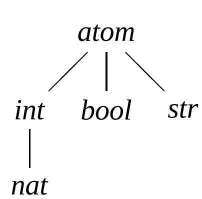
$$\begin{array}{l} \textbf{let } f_1 = \Lambda\alpha <: t_{i_1}.\lambda x : \alpha.c_1\ x \textbf{ in} \\ \quad\vdots \\ \textbf{let } f_n = \Lambda\alpha <: t_{i_n}.\lambda x : \alpha.c_n\ x \textbf{ in} \\ \quad e \end{array}$$

By Theorem 5.4, the translation of the above program executes without run-time errors. Furthermore, by Theorem 5.3, the phantom types encoding of the types of these constants are sound with respect to δ in $\lambda^{\mathrm{DM}}_{\mathsf{T}}$. Hence, by Theorem 5.2, $\lambda^{\mathrm{DM}}_{\mathsf{T}}$ with these constants is sound and we can safely use these constants in $\lambda^{\mathrm{DM}}_{\mathsf{T}}$. Therefore, we can replace to the body of the translated program with an arbitrary $\lambda^{\mathrm{DM}}_{\mathsf{T}}$ expression that type-checks in that context and the resulting program will still execute without run-time errors. Essentially, the translation of the let bindings corresponds to a "safe" interface to the primitives; programs that use this interface in a type-safe manner are guaranteed to execute without run-time errors.

5.4 Example and remarks

In this section, we work through a mostly complete example before turning our attention to some general remarks.

Recall the subtyping hierarchy introduced in Section 2, extended in Section 4, and here further extended to include strings.



We can encode this hierarchy with phantom types as follows:

$$\begin{aligned}
\langle \mathit{atom} \rangle_A &= \alpha \times (\beta \times \gamma) & \langle \mathit{atom} \rangle_C &= 1 \times (1 \times 1) \\
\langle \mathit{int} \rangle_A &= \mathsf{T}\,\alpha \times (\beta \times \gamma) & \langle \mathit{int} \rangle_C &= \mathsf{T}\,1 \times (1 \times 1) \\
\langle \mathit{nat} \rangle_A &= \mathsf{T}\,(\mathsf{T}\,\alpha) \times (\beta \times \gamma) & \langle \mathit{nat} \rangle_C &= \mathsf{T}\,(\mathsf{T}\,1) \times (1 \times 1) \\
\langle \mathit{bool} \rangle_A &= \alpha \times (\mathsf{T}\,\beta \times \gamma) & \langle \mathit{bool} \rangle_C &= 1 \times (\mathsf{T}\,1 \times 1) \\
\langle \mathit{str} \rangle_A &= \alpha \times (\beta \times \mathsf{T}\,\gamma) & \langle \mathit{str} \rangle_C &= 1 \times (1 \times \mathsf{T}\,1)
\end{aligned}$$

We consider two primitive operations `double` and `toString` with

$$\begin{aligned}
\pi_p(\mathtt{double}) &= \{\mathit{int} \to \mathit{int},\ \mathit{nat} \to \mathit{nat}\} \\
\pi_p(\mathtt{toString}) &= \{\mathit{atom} \to \mathit{str},\ \mathit{int} \to \mathit{str},\ \mathit{nat} \to \mathit{str},\ \mathit{bool} \to \mathit{str},\ \mathit{str} \to \mathit{str}\}
\end{aligned}$$

We can derive the following typing judgements in $\lambda^{\mathrm{DM}}_{<:}$, which capture the intended subtyping:

$$\begin{aligned}
&\vdash_{<:} \Lambda\alpha <: \mathit{int}.\lambda x : \alpha.\mathtt{double}\, x : \forall\alpha <: \mathit{int}.\alpha \to \alpha \\
&\vdash_{<:} \Lambda\alpha <: \mathit{atom}.\lambda x : \alpha.\mathtt{toString}\, x : \forall\alpha <: \mathit{atom}.\alpha \to \mathtt{str}
\end{aligned}$$

Applying our translation to these functions yields:

$$\begin{aligned}
\mathcal{E}[\![\Lambda\alpha <: \mathit{int}.\lambda x : \alpha.\mathtt{double}\, x]\!] &= \Lambda\alpha,\beta,\gamma.\lambda x : \mathsf{T}\,(\mathsf{T}\,\alpha \times (\beta \times \gamma)).\mathtt{double}\, x \\
\mathcal{E}[\![\Lambda\alpha <: \mathit{atom}.\lambda x : \alpha.\mathtt{toString}\, x]\!] &= \Lambda\alpha,\beta,\gamma.\lambda x : \mathsf{T}\,(\alpha \times (\beta \times \gamma)).\mathtt{toString}\, x
\end{aligned}$$

As expected from Theorem 5.4, we can derive typing judgements that assign the translated types to these functions:

$$\begin{aligned}
&\vdash_{\mathsf{T}} \Lambda\alpha,\beta,\gamma.\lambda x : \mathsf{T}\,(\mathsf{T}\,\alpha \times (\beta \times \gamma)).\mathtt{double}\, x : \forall\alpha,\beta,\gamma.\mathsf{T}\,(\mathsf{T}\,\alpha \times (\beta \times \gamma)) \to \mathsf{T}\,(\mathsf{T}\,\alpha \times (\beta \times \gamma)) \\
&\vdash_{\mathsf{T}} \Lambda\alpha,\beta,\gamma.\lambda x : \mathsf{T}\,(\alpha \times (\beta \times \gamma)).\mathtt{toString}\, x : \forall\alpha,\beta,\gamma.\mathsf{T}\,(\alpha \times (\beta \times \gamma)) \to \mathsf{T}\,(1 \times (1 \times \mathsf{T}\,1))
\end{aligned}$$

Interestingly, we can also derive the following typing judgements:

$$\begin{aligned}
&\vdash_{\mathsf{T}} \Lambda\alpha.\lambda x : \mathsf{T}\,(\mathsf{T}\,\alpha \times (\alpha \times \alpha)).\mathtt{double}\, x : \forall\alpha.\mathsf{T}\,(\mathsf{T}\,\alpha \times (\alpha \times \alpha)) \to \mathsf{T}\,(\mathsf{T}\,\alpha \times (\alpha \times \alpha)) \\
&\vdash_{\mathsf{T}} \Lambda\alpha,\beta.\lambda x : \mathsf{T}\,(\alpha \times (\beta \times \beta)).\mathtt{toString}\, x : \forall\alpha,\beta.\mathsf{T}\,(\alpha \times (\beta \times \beta)) \to \mathsf{T}\,(1 \times (1 \times \mathsf{T}\,1))
\end{aligned}$$

The first function type-checks because, of all basic types, only $\mathsf{T}\ \langle \mathit{int} \rangle_C$ unifies with $\mathsf{T}\ (\mathsf{T}\ \alpha \times (\alpha \times \alpha))$, by the substitution $(\alpha, 1)$, and $\{\mathsf{T}\ \langle \mathit{int} \rangle_C \to \mathsf{T}\ \langle \mathit{int} \rangle_C\} \subseteq \mathcal{T}[\![\pi_p]\!](\mathtt{double})$. Likewise, the second function type-checks because, of all basic types, only $\mathsf{T}\ \langle \mathit{value} \rangle_C$, $\mathsf{T}\ \langle \mathit{int} \rangle_C$, $\mathsf{T}\ \langle \mathit{nat} \rangle_C$ unify with $\mathsf{T}\ (\mathsf{T}\ \alpha \times (\beta \times \beta))$ and $\{\mathsf{T}\ \langle \mathit{atom} \rangle_C \to \mathsf{T}\ \langle \mathit{str} \rangle_C,\ \mathsf{T}\ \langle \mathit{int} \rangle_C \to \mathsf{T}\ \langle \mathit{str} \rangle_C,\ \mathsf{T}\ \langle \mathit{nat} \rangle_C \to \mathsf{T}\ \langle \mathit{str} \rangle_C\} \subseteq \mathcal{T}[\![\pi_p]\!](\mathtt{toString})$. We can interpret the first as a function that can only be applied to integers (but not naturals) and the second as a function that can only be applied to values, integers, and naturals (but not booleans or strings). Observe that while these functions do not capture all of the subtyping available in their wrapped primitive operations, neither do they violate the subtyping available. This corresponds to the fact that the second set of types are instances of the first set of types under appropriate substitutions for β and γ.

The existence of these typing judgements sheds some light on the practical aspects of using the phantom types technique in real programming languages. Recall that the typing judgement for primitive operations is somewhat non-standard. Specifically, in contrast to most typing judgements for primitives (like the typing judgement for basic constants), this judgement is not syntax directed;

that is, the type is not uniquely determined by the primitive operation. This complicates a type-inference system for $\lambda_{\mathsf{T}}^{\mathrm{DM}}$. At the same time, we cannot expect to integrate this typing judgement into an existing language with a Hindley-Milner style type system. Rather, we expect to integrate a primitive operation into a programming language through a foreign-function interface, at which point we give the introduced function a very basic type that does not reflect the subtyping inherent in its semantics.[4] After introducing the primitive operation in this fashion, we wrap it with a function to which we can assign the intended type using the phantom types encoding, because the type system will not, in general, infer the appropriate type. It is for this reason that we have stressed the application of phantom types technique to developing and implementing interfaces.

6 An application to datatype specialization

Let us say a datatype is *specialized* when we define a version of the datatype with a subset of its constructors, themselves acting on specializations of the datatype. If we view elements of a datatype as data structures, elements of the specialized datatype are data structures obeying certain restrictions. For example, a representation of boolean formulas can be specialized to represent formulas in disjunctive normal form. A representation of lists can be specialized to distinguish between the empty list and nonempty lists, or even lists with a given number of elements. By choosing appropriate datatype definitions and specializations, we can capture a variety of program invariants. By using a type system that is aware of such invariants, we can use typechecking to verify that these invariants are preserved; compile-time type errors will indicate errors that could violate program invariants. A recent type system that enforces similar (and strictly more powerful) invariants is the *refinement types* system [Freeman and Pfenning 1991; Freeman 1994].

Let's introduce a real example to make this discussion more concrete. Consider the boolean formulas mentioned earlier, which we will use as a running example throughout this section. The first problem is defining an abstract syntax for formulas, of which $p \wedge (\texttt{true} \vee \neg q)$ is a typical example. A straightforward representation of formulas is the following:

```
datatype fmla = Var of string | Not of fmla
              | True | And of fmla * fmla
              | False | Or of fmla * fmla
```

We can easily define a function `eval` that takes a formula and an environment associating every variable in the formula with a truth value, and evaluates the formula. Similarly, we can define a `toString` function that takes a formula and returns a string representation of the formula. As is well known, a propositional formula can always be represented in a special form called Disjunctive Normal Form (or DNF), as a disjunction of conjunctions of variables and literals. A formula in DNF is still a formula, but it has a restricted form. We can define a DNF formula as a *specialization* of the above formulas. We assume a special syntax for specializations, which should be self-explanatory:

```
datatype fmla   = Var of string | Not of fmla
                | True | And of fmla * fmla
                | False | Or of fmla * fmla
```

[4]In general, foreign-function interfaces have strict requirements on the types of foreign functions that can be called. Due to internal implementation details, language implementations rarely allow foreign functions to be given polymorphic types or types with user defined datatypes, both of which are used by the phantom types encodings.

```
withspec atom = Var of string
     and lit  = Var of string | Not of atom
     and conj = True | And of lit * conj
     and dnf  = False | Or of conj * dnf
```

Roughly speaking, the specialization `dnf` of the datatype `fmla` is restricted so that the `Or` constructor creates list of conjunctions terminated with the `False` constructor. A conjunction is defined by another specialization `conj` of the datatype `fmla` that restricts the `And` constructor to forming lists of literals. A literal is essentially a variable or a negated variable. This can be captured using two specializations, `atom` for atomic literals and `lit` for literals. Notice that to define the `dnf` specialization, we need all the specializations `dnf`, `conj`, `lit`, and `atom`.

In this section, we show that we can implement, in SML, much of what one would want out of a language that directly supports specialization through the type system, such as a refinement type system. We can write, for example, a function `toDnf` that guarantees not only that its result is a formula, but also that it is in DNF form, purely statically. The advantage of expressing specialization invariants directly in SML, of course, is that the former type systems are complex and not widely available. What are the key features of specialization that we would like available? For one, we would like the representation of values of specialized types to be the same as the representation of the original datatype. Hence, we should be able to implement a single function to evaluate not only an unspecialized formula, but also any specialization of formulas, such as the `dnf` specialization. Moreover, we would like to write `case` expressions that do not include branches for constructors that do not occur in the specialization of the value being examined. For example, if we perform a case analysis on a value with specialization `dnf`, we should only need to supply a branch for the `False` and `Or` constructors. If we write such an expression in SML, we are warned that the `case` expression is not exhaustive. Clearly, we could accomplish the above by having distinct datatypes for formulas and DNF formulas and provide functions to explicitly convert between them. This is of course inefficient, and, as we show, unnecessarily so.

We exhibit an informal translation based on phantom types from a set of specializations of a datatype to declarations providing constructors, destructors and coercions corresponding to the specializations. These declarations form a minimal set of primitive operations that provide the functionality of the specialization. They can be used in SML to enforce the invariants specified by the specializations. While the translation generates a large amount of code, most of this code is boilerplate code that can be mechanically generated, yielding appropriate structures and signatures providing access to the specialized types.

Our notion of specialization is quite general, and can capture a good number of useful invariants. For example, we can define a specialization that ensures that the formula contains no variables:

```
datatype fmla = Var of string | Not of fmla
              | True | And of fmla * fmla
              | False | Or of fmla * fmla
  withspec grnd = Not of grnd
                | True | And of grnd * grnd
                | False | Or of grnd * grnd
```

The following specializations of a Peano number representations distinguish between zero and non-zero:

```
datatype peano = Zero | Succ of peano
  withspec zero = Zero
       and nonzero = Succ of peano
```

Lists can give rise to interesting specializations. Consider the following specializations, differentiating between empty lists, singleton lists, and nonempty lists:

```
datatype 'a list = Nil | Cons of 'a * 'a list
  withspec 'a empty = Nil
       and 'a singleton = Cons of 'a * 'a empty
       and 'a nonempty = Cons of 'a * 'a list
```

The following list specializations distinguish between lists of even or odd length:

```
datatype 'a list = Nil | Cons of 'a * 'a list
  withspec 'a even = Nil | Cons of 'a * 'a odd
       and 'a odd = Cons of 'a * 'a even
```

Finally, we can use specializations to define abstract syntax trees that distinguish between arbitrary expressions and well-formed expressions (e.g., well-typed expressions [Leijen and Meijer 1999; Elliott, Finne, and de Moor 2000], expressions in normal forms, etc.). A simple example of this is the following:

```
datatype exp = Bool of bool
             | And of exp * exp
             | Int of int
             | Plus of exp * exp
             | If of exp * exp * exp
  withspec boolexp
             = Bool of bool
             | And of boolexp * boolexp
             | If of boolexp * boolexp * boolexp
       and intexp
             = Int of int
             | Plus of intexp * intexp
             | If of boolexp * intexp * intexp
```

Using this form of datatype specialization, we have show how to define red-black trees that check the critical invariant that no red node has a red child child after inserting a new element and how to define constructors for expressions in the simply-typed λ-calculus that permit only the building of type-correct expressions.

6.1 Writing specializations in SML

How should we write (in SML) an implementation of the above formula dataype and its specializations, so that the type system enforces a consistent use of constructors? In this section, we give a highly-stylized implementation that achieves this particular goal. We hope that the reader will grasp

```
signature FMLA = sig
  (* specialization type *)
  type 'a t

  (* encoding types *)
  datatype 'a x = X and 'a y = Y and 'a z = Z

  (* abstract encodings *)
  type 'a AFmla = 'a t
  type 'a ALit = ('a x) t
  type 'a AAtom = (('a x) x) t
  type 'a AConj = ('a y) t
  type 'a ADnf = ('a z) t

  (* concrete encodings *)
  type CFmla = unit t
  type CLit = (unit x) t
  type CAtom = ((unit x) x) t
  type CConj = (unit y) t
  type CDnf = (unit z) t

  (* constructors *)
  val varFmla : string -> CFmla
  val notFmla : 'a AFmla -> CFmla
  val trueFmla : CFmla
  val andFmla : 'a AFmla * 'b AFmla -> CFmla
  val falseFmla : CFmla
  val orFmla : 'a AFmla * 'b AFmla  -> CFmla
  val varLit : string -> CLit
  val notLit : 'a AAtom -> CLit
  val varAtom : string -> CAtom
  val trueConj : CConj
  val andConj : 'a ALit * 'b AConj -> CConj
  val falseDnf : CDnf
  val orDnf : 'a AConj * 'b ADnf  -> CDnf

  (* destructors *)
  val caseFmla : 'a AFmla -> (string -> 'b) * (CFmla -> 'b) *
                             (unit -> 'b) * (CFmla * CFmla -> 'b) *
                             (unit -> 'b) * (CFmla * CFmla -> 'b) -> 'b
  val caseLit : 'a ALit -> (string -> 'b) * (CAtom -> 'b) -> 'b
  val caseAtom : 'a AAtom -> (string -> 'b) -> 'b
  val caseConj : 'a AConj -> (unit -> 'b) * (CLit * CConj -> 'b) -> 'b
  val caseDnf : 'a ADnf -> (unit -> 'b) * (CConj * CDnf -> 'b) -> 'b

  (* coercions *)
  val coerceFmla : 'a AFmla -> CFmla
  val coerceLit : 'a ALit -> CLit
  val coerceAtom : 'a AAtom -> CAtom
  val coerceConj : 'a AConj -> CConj
  val coerceDnf : 'a ADnf -> CDnf
end
```

Figure 4: The FMLA signature

```
signature FMLA = sig
  (* specialization type *)
  type 'a t

  (* encoding types *)
  datatype 'a x = X and 'a y = Y and 'a z = Z

  (* abstract encodings *)
  type 'a AFmla = 'a t
  type 'a ALit = ('a x) t
  type 'a AAtom = (('a x) x) t
  type 'a AConj = ('a y) t
  type 'a ADnf = ('a z) t

  (* concrete encodings *)
  type CFmla = unit t
  type CLit = (unit x) t
  type CAtom = ((unit x) x) t
  type CConj = (unit y) t
  type CDnf = (unit z) t

  (* constructors *)
  val varFmla : string -> CFmla
  val notFmla : 'a AFmla -> CFmla
  val trueFmla : CFmla
  val andFmla : 'a AFmla * 'b AFmla -> CFmla
  val falseFmla : CFmla
  val orFmla : 'a AFmla * 'b AFmla  -> CFmla
  val varLit : string -> CLit
  val notLit : 'a AAtom -> CLit
  val varAtom : string -> CAtom
  val trueConj : CConj
  val andConj : 'a ALit * 'b AConj -> CConj
  val falseDnf : CDnf
  val orDnf : 'a AConj * 'b ADnf  -> CDnf

  (* destructors *)
  val caseFmla : 'a AFmla -> (string -> 'b) * (CFmla -> 'b) *
                             (unit -> 'b) * (CFmla * CFmla -> 'b) *
                             (unit -> 'b) * (CFmla * CFmla -> 'b) -> 'b
  val caseLit : 'a ALit -> (string -> 'b) * (CAtom -> 'b) -> 'b
  val caseAtom : 'a AAtom -> (string -> 'b) -> 'b
  val caseConj : 'a AConj -> (unit -> 'b) * (CLit * CConj -> 'b) -> 'b
  val caseDnf : 'a ADnf -> (unit -> 'b) * (CConj * CDnf -> 'b) -> 'b

  (* coercions *)
  val coerceFmla : 'a AFmla -> CFmla
  val coerceLit : 'a ALit -> CLit
  val coerceAtom : 'a AAtom -> CAtom
  val coerceConj : 'a AConj -> CConj
  val coerceDnf : 'a ADnf -> CDnf
end
```

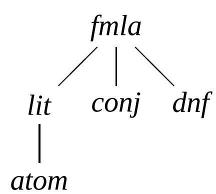
Figure 5: The `Fmla` structure

the straightforward generalization of this implementation to an arbitrary specialization.[5] We feel that a fully elaborated example is more instructive than a formal translation, where definitions and notation become burdensome and obfuscating.

Consider the `fmla` datatype and the `dnf` specialization introduced at the beginning of this section:

```
datatype fmla   = Var of string | Not of fmla
                | True | And of fmla * fmla
                | False | Or of fmla * fmla
  withspec atom = Var of string
       and lit  = Var of string | Not of atom
       and conj = True | And of lit * conj
       and dnf  = False | Or of conj * dnf
```

The specializations induce a simple subtyping relationship between the various kinds of formulas:



For uniformity, we will consider the specialized datatype itself as a specialization.

Figures 4 and 5 give a signature and corresponding implementation of the specializations above. The amount of code may seem staggering for such a small example, but as we shall see shortly, most of it is boilerplate code. (In fact, it is straightforward to mechanically generate this code from a declarative description of the specializations.) Moreover, all the action is in the signature! The implementation is trivial. Part of the reason for this explosion of code is that while SML provides convenient syntax for datatypes, there is no syntax for specialization. Therefore, we implement a datatype and its specializations notionally as abstract types with explicit constructors and destructors. In other words, we give a datatype

```
datatype t = A of string | B of int
```

the following interface:

```
type t
val aT : string -> foo
val bT : int -> foo
val caseT : t ->
            (string -> 'b) *
            (int -> 'b) -> 'b
```

[5] Our running example uses a first-order, monomorphic datatype, of which abstract syntax trees are a typical example. Extending the implementation to handle first-order, polymorphic datatypes is straightforward. It is also possible to handle higher-order datatypes; we briefly consider this in Section 6.3.

where aT and bT are used to construct values of type t and caseT is used to perform a case analysis on a value of type t that invokes the appropriate supplied function on the constructor argument. It's easy to see how the blowup from a single-line datatype declaration to a four-line description of an abstract interface leads to the kind of interface we have in Figure 4. With this in mind, let us examine the different elements of the signature and their implementation.

6.1.1 Types

The first part of the signature defines the types for formulas and their specializations. This uses the phantom types approach described in the first part of this paper. We introduce a polymorphic type t. This type represents the values of the datatype and its specializations. This ensures that these values all have the same representation.

To differentiate between the different specializations, we define a series of type abbreviations CFmla, CAtom, CLit, CConj, and CDnf, which encode the *concrete types*. A value of type CAtom will correspond to a value in the atom specialization. Concrete types are abbreviations of the t type constructor, applied to a particular type that represents the specialization. For instance, the type CLit is declared as

```
type CLit = (unit x) t
```

Here, unit x is the type corresponding to the specialization lit in our encoding of the subtyping hierarchy given above. To help in the encoding, the signature introduces the types x, y, and z. (This is essentially the tree hierarchies encoding of Section 3.1.)

As we mentioned, there is an implicit subtyping hierarchy on specializations. To capture subtyping, corresponding to every concrete type, we define a type that can match every subtype of that concrete type. Hence, we define AFmla, AAtom, ALit, AConj, and ADnf, which encode the *abstract types*. These types allow us to write, for instance, a function of type 'a ALit → string, which takes an arbitrary subtype of the specialization lit, and returns a string representation of that formula. To see why this works, look at the definition of the type ALit:

```
type 'a ALit = ('a x) t
```

We can verify that indeed, CLit and CAtom match ALit. We can also verify that all concrete types match AFmla, capturing the fact that the fmla specialization is the top element of the subtyping hierarchy. In other words, the encoding respects the subtyping hierarchy on specializations.

In the implementation of Figure 5, we see that the type t is implemented as a *bona fide* SML datatype whose polymorphic type variable is ignored. Hence, all the specializations share the same representation. However, notice that we use *opaque signature matching* in Figure 5. This is crucial to get the required behavior for the phantom types, as we saw in Section 2.[6]

6.1.2 Constructors

For every specialization, the interface provides a function for each constructor of the specialization. For instance, the atom specialization has a single constructor Var, so we provide a function

[6]For languages that do not provide opaque signature matching, such as Haskell, this particular way of hiding phantom types does not work. For those, we need to make t a datatype with a dummy constructor, and wrap and unwrap every value with that dummy constructor.

```
val varAtom : string -> CAtom
```

that returns an element of the specialization `atom` (and hence, of type `CAtom`). We allow subtyping on the constructor arguments, where appropriate. Hence, the constructor `And` for `conj` is available as:

```
val andConj : 'a ALit * 'b AConj -> CConj
```

The implementation of these constructors is trivial. They are simply abbreviations for the actual constructors of the datatype `t`. Giving them different names allow us to constrain their particular type, depending on the specialization we want them to yield.

6.1.3 Destructors

For every specialization, the interface provides a destructor function that can be used to simultaneously discriminate and deconstruct elements of the specialization, similar to the manner in which the `case` expression operates in SML. Each destructor function takes an element of the specialization as well as functions to be applied to the arguments of the matched constructor.

As an example, let's examine the `caseFmla` function, the destructor function for the specialization `fmla`. It has the following type:

```
val caseFmla : 'a AFmla ->
               (string -> 'b) *
               (CFmla -> 'b) *
               (unit -> 'b) *
               (CFmla * CFmla -> 'b) *
               (unit -> 'b) *
               (CFmla * CFmla -> 'b) -> 'b
```

It expects first a value of any subtype of `fmla`; therefore, we use the abstract type `AFmla`. Because the `fmla` specialization has constructors `Var`, `Not`, `True`, `And`, `False`, and `Or`, we must provide functions to apply in each of these cases. For example, if the value passed to `caseFmla` was constructed using `Var`, then the first function is applied to the string argument of `Var`. If the value passed was constructed using `Not`, then the second function is applied to the `fmla` argument of `Not`. Similarly for the other cases.

One may note that when the constructor to be matched has arguments which are themselves specializations, a concrete type is used for the function. For example, the function used to match a `Not` constructor in `caseFmla` has type `CFmla` → `'b`. This seems limiting. We might expect it to take a function `'c AFmla` → `'b` instead, since presumably that function ought to be able to handle any subtype of `fmla`. Unfortunately, doing this would break the invariants of the specializations. Consider the following expression, which is well-typed when the `Not` branch of `caseFmla` is given the alternate type `'c AFmla` → `'b`:

```
val res =
  caseFmla (notFmla (notFmla (varFmla "p")))
    (fn _ => varLit "q",
     fn f => notLit f,
     fn _ => varLit "q",
```

```
      fn _ => varLit "q",
      fn _ => varLit "q",
      fn _ => varLit "q")
```

This expression has the type `CLit`, but it evaluates to the formula `Not (Not (Var "p"))`, which is not a valid element of the specialization `lit`. The problem is that the anonymous function `fn f => notLit f` has the type `'a ALit` $\rightarrow$ `CLit`, which can be unified with `'c AFmla` $\rightarrow$ `'b` by taking `'c` $\leftarrow$ `'a x` and `'b` $\leftarrow$ `CLit`.

Informally, the invariants of the specializations are preserved when we use concrete encodings in covariant type positions and abstract encodings in contravariant type positions. This explains the appearance of concrete encodings in the argument types of the branch functions.

As a different example, consider the destructor function `caseConj` for the specialization `conj`. Because the elements of `conj` are built using only the `True` and `And` constructors, deconstructing elements of such a specialization can only yield the `True` constructor or the `And` constructor applied to a `lit` value and to a `conj` value. Therefore, we give `caseConj` the type:

```
val caseConj : 'a AConj ->
               (unit -> 'b) *
               (CLit * CConj -> 'b) -> 'b
```

Similar reasoning allows us to drop or refine the types of various branches in the destructor functions for the other specializations.

Destructor functions also have a trivial implementation. They are simply implemented as a SML `case` expression. On the branches for which no function is provided, we raise an exception. On the other hand, if the invariants of the specializations are enforced, we know that those exceptions will never be raised! By virtue of our encoding of subtyping, static typing ensures that this exception will never be raised by programs that use the interface. (This essentially follows from our results of Section 5.)

6.1.4 Coercions

Finally, the interface provides coercion functions that convert subtypes to supertypes. Such coercion functions are necessary because, intuitively, phantom types provide only a restricted form of subtyping. For instance, type subsumption occurs only at type application, which most often coincides with function application. Thus, when two expressions of different specializations occur in contexts that must have equal types, such as the branches of an `if` expression, subsumption does not occur, and the expressions must be coerced to a common supertype. For example, the following function will not typecheck because the type of the true branch is `CLit` and the type of the false branch is `CAtom` – two types that cannot be unified.

```
fun bad b = if b
              then varLit ("p")
              else varAtom ("q")
```

Instead, we must write the following, coercing the `else` branch to the common supertype `CLit`:

```
fun good b = if b
               then varLit ("p")
               else coerceLit (varAtom ("q"))
```

Coercions are also useful to work around a restriction in SML that precludes the use of polymorphic recursion [Henglein 1993; Kfoury, Tiuryn, and Urzyczyn 1993]. We shall shortly see an example where this use of a coercion is necessary.

The implementation of coercion functions is trivial. They are simply identity functions that change the type of a value.

6.2 Examples

Let us give a few examples of functions that can be written against the interface to formula specializations given above.

First, consider a `toString` function that returns a string representation of a formula. A simple implementation is the following:

```
fun toString (f: 'a AFmla): string = let
  fun toString' (f: CFmla) =
    caseFmla f
      (fn s => s,
       fn f => concat ["-", toString' f],
       fn () => "T",
       fn (f1,f2) => concat ["(", toString' f1, " & ",
                             toString' f2, ")"],
       fn () => "F",
       fn (f1,f2) => concat ["(", toString' f1, " | ",
                             toString' f2, ")"])
in toString' (coerceFmla f) end
```

Note that the inferred type of the `toString'` function is `CFmla` → `string`, because it is recursively applied to variables of type `CFmla` in the `caseFmla` branches and SML does not support polymorphic recursion. However, we can recover a function that allows subtyping on its argument by composing `toString'` with an explicit coercion. Now the SML type system infers the desired type for the `toString` function.

If we had polymorphic recursion, we could directly assign the type `'a AFmla` → `string` to `toString'`. One may ask whether the lack of polymorphic recursion in SML poses a significant problem for the use of specializations as we have described them. The fact that we could work around the problem in this one instance does not mean that another example would not need polymorphic recursion in an essential way. However, this is not the case. This relies on the fact that the use of phantom types in specializations only influences the *type* of an expression or value, never its representation. The specialization idiom only guarantees certain structural invariants. Consider a recursive function which accepts an argument of a certain specialization *sp*. (This generalizes to multiple arguments in a straightforward way.) Any recursive call that it makes sense to perform must be applied to a subtype of *sp*; furthermore, there exists a coercion from that subtype to *sp*. Hence, we can always write a recursive function as we did above: set the domain of the function to the concrete encoding of *sp* and coerce to *sp* at any recursive call. (In the `toString` example, all of the recursive calls are on values of type `CFmla`, so the coercion can be elided.) We recover the subtyping of the original function by coercing the argument to *sp* prior to calling the recursive

```
fun andConjs (f: CConj, g: CConj): CConj =
  caseConj f (fn () => g, fn (f1,f2) => andConj (f1, andConjs (f2, g)))
fun orDnfs (f: CDnf, g: CDnf): CDnf =
  caseDnf f (fn () => g,  fn (f1,f2) => orDnf (f1, orDnfs (f2, g)))
fun andConjDnf (f: CConj, g: CDnf): CDnf =
  caseDnf g (fn () => falseDnf,
             fn (g1,g2) => orDnf (andConjs (f, g1), andConjDnf (f, g2)))
fun andDnfs (f: CDnf, g: CDnf): CDnf =
  caseDnf f
    (fn () => falseDnf,
     fn (f1,f2) => caseDnf g
                     (fn () => falseDnf,
                      fn (g1,g2) => orDnf (andConjs (f1, g1),
                                    orDnfs (andConjDnf (f1, g2),
                                    orDnfs (andConjDnf (g1, f2),
                                            andDnfs (f2, g2))))))
fun litToDnf (f: 'a ALit): CDnf = orDnf (andConj (f, trueConj), falseDnf)
fun toDnf (f: 'a AFmla): CDnf = let
  fun toDnf' (f: CFmla): CDnf =
    caseFmla f
      (fn s => litToDnf (varAtom s),
       fn f => caseFmla f
                 (fn s => litToDnf (notLit (varAtom s)),
                  fn f => toDnf' f,
                  fn () => toDnf' falseFmla,
                  fn (f,g) => toDnf' (orFmla (notFmla f, notFmla g)),
                  fn () => toDnf' trueFmla,
                  fn (f,g) => toDnf' (andFmla (notFmla f, notFmla g))),
       fn () => orDnf (trueConj, falseDnf),
       fn (f,g) => andDnfs (toDnf' f, toDnf' g),
       fn () => falseDnf,
       fn (f,g) => orDnfs (toDnf' f, toDnf' g))
in toDnf' (coerceFmla f) end
```

Figure 6: The `toDnf` function

function.[7]

Figure 6 gives an extended example which culminates with a `toDnf` function that converts any formula into an equivalent DNF formula. The type of this function, `'a AFmla` $\rightarrow$ `CDnf`, ensures that the result formula is a DNF formula. We further note that the use of type annotations in Figure 6 is completely superfluous. Type-inference will deduce precisely these types.

A typical problem that arises when providing constructors and destructor functions in place of a proper datatype declaration is that we lose the ability to perform pattern-matching on the values of the type. Using ideas from Wang and Murphy [2003], we show in Appendix E that we can in fact recover pattern-matching for our implementation of datatype specializations. Therefore, specializations can be used with practically no overhead from the programmer.

6.3 Discussion

In our previous discussion, we have made a number of implicit and explicit restrictions to simplify the treatment of specializations. There are a number of ways of relaxing these restrictions that

[7] In this explication, we assume that the desired recursive function on the unspecialized datatype can itself be written without polymorphic recursion. If this is not the case, then the function cannot be written in SML with or without phantom types and specializations. (Thus, we exclude specializations of non-regular datatypes.) On the other hand, if this is the case, then one need never "escape" from specializations to write the function.

result in more expressive systems. For example, we can allow a specialization to use the same tag at multiple argument types. Consider defining a DNF formula to be a list of Or-ed conjuctions that grows to either the left or the right:

```
withspec dnf = False
             | Or of conj * dnf
             | Or of dnf * conj
```

One can easily define two constructor functions for Or that inject into the dnf specialization. However, the "best" destructor function that one can write is the following:

```
val caseDnf : 'a ADnf ->
              (unit -> 'b) *
              (CFmla * CFmla -> 'b) -> 'b
```

not, as might be expected:

```
val caseDnf : 'a ADnf ->
              (unit -> 'b) *
              (CConj * CDnf -> 'b) *
              (CDnf * CConj -> 'b) -> 'b
```

While the second function can be written with the expected semantics, it requires a run-time inspection of the arguments to the Or constructor to distinguish between a conj and a dnf. We do not consider this implementation to be in the spirit of a primitive case expression, because sufficiently complicated specializations could require non-constant time to execute a destructor function. Instead, the first function corresponds to the least-upper bound of conj $\times$ dnf and dnf $\times$ conj in the upper semi-lattice of types induced by the specialization subtyping hierarchy. The loss of precision in the resulting type corresponds to the fact that no specialization exactly corresponds to the union of the conj and dnf specializations. We could gain some precision by introducing such a specialization, at the cost of complicating the interface.

In the examples discussed previously, we have restricted ourselves to monomorphic, first-order datatypes. This restiction can be relaxed to allow monomorphic, higher-order datatypes, although this extension requires the specialization subtyping hierarchy to induce a full lattice of types (rather than an upper semi-lattice), due to the contravariance of function arguments.

One final limitation of the procedure described here is that it only applies to one datatype. At times, it may be desirable to consider specializations of one datatype in terms of the specializations of another datatype. The technique described in this paper can be extended to handle this situation by processing all of the specialized datatypes simultaneously. Although this decreases the modularity of a project, it is required to declare each unspecialized datatype in terms of other unspecialized datatypes, which otherwise would be hidden by the opaque signatures.

As we already mentioned, a system that enforces invariants similar in spirit to (and strictly more expressive than) specializations is the refinement types system [Freeman and Pfenning 1991; Freeman 1994]. In fact, many of the examples considered here were inspired by similar examples expressed in the refinement types setting. However, there are a number of critical differences between refinement types and the specialization techniques described in this paper. These differences permit us to express specializations directly within SML's type system. The most significant difference concerns the "number" of types assigned to a value. In short, refinement types use a limited form of type intersection to assign a value multiple types, each corresponding to the evaluation of the value at specific refinements, while specialization assigns every value exactly one type.[8] For example, in Section 6.1, we assigned the `litToDnf` function the (conceptual) type `lit` $\rightarrow$ `dnf`. In the refinements type setting, it would be assigned the type (`fmla` $\rightarrow$ `fmla`) $\wedge$ (`lit` $\rightarrow$ `dnf`), indicating that in addition to mapping literals to DNF formulae, the function can also be applied to an arbitrary formula, although the resulting formula will not satisfy any of the declared refinements.

While this demonstrates the expressibility of the refinement types system, it does not address the utility of this expressiveness. In particular, one rarely works in a context where *all* possible typings of an expression are necessary. In fact, the common case, particularly with data structure invariants, is a context where exactly one type is of interest – namely, that a "good" structure is either produced or preserved. This is exactly the situation that motivates our examples of specialization. In this sense, specialization is closer in spirit to refinement type checking [Davies 1997], which verifies that an expression satisfies the user specified refinement types.

It is worth pointing out that some limitations of our implementation of specializations are due to the encodings of the subtyping hierarchy implicit in the specializations. So long as the encodings respects the hierarchy, the techniques described in this paper are completely agnostic to the specifics of the encoding. However, if the encoding of the subtyping hierarchy has other properties beyond respecting the hierarchy, these properties can sometimes be used to provide a more flexible implementation of specializations. The following example should give a flavor of the kind of flexibility we have in mind.

Consder a datatype of bit strings, and specializations that capture the parity of bit strings:

```
datatype bits   = Nil
                | Zero of bits
                | One of bits
  withspec even = Nil
                | Zero of even
                | One of odd
       and odd  = Zero of odd
                | One of even
```

Following the approach described in this section, it is straightforward to derive an interface to bit strings and their specializations:

```
signature BITS = sig
  (* specialization and encoding types *)
  type 'a t
  type 'a x and 'a y

  (* abstact and concrete encodings *)
  type 'a ABits = 'a t
  type 'a AEven = ('a x) t
  type 'a AOdd = ('a y) t
  type CBits = unit ABits
  type CEven = unit AEven
```

[8] Although, both systems employ a form of subtyping to further increase the "number" of types assigned to a value.

```
    type COdd = unit AOdd

    (* constructors *)
    val nilEven : CEven
    val zeroEven : 'a AEven -> CEven
    val oneEven : 'a AOdd -> CEven
    val zeroOdd : 'a AOdd -> COdd
    val oneOdd : 'a AEven -> COdd
    ...
  end
```

An analysis of the above interface reveals that we can in fact define a single `zero` constructor that applies to all specializations, with type `'a t` $\rightarrow$ `'a t`. Unfortunately, we cannot similarly define a single `one` constructor.

However, if we choose the encoding of the subtyping hierarchy carefully, we can in fact come up with an implementation of the bit strings specializations that allows for the definition of a single `one` constructor:

```
  signature BITS = sig
    (* specialization and encoding types *)
    type 'a t
    type 'a z

    (* abstact and concrete encodings *)
    type ('a, 'b) ABits = ('a * 'b) t
    type ('a, 'b) AEven = ('a z * 'b) t
    type ('a, 'b) AOdd = ('a * 'b z) t
    type CBits = (unit, unit) ABits
    type CEven = (unit, unit) AEven
    type COdd = (unit, unit) AOdd

    (* constructors *)
    val nilEven : CEven
    val zero : ('a, 'b) t -> ('a, 'b) t
    val one : ('a, 'b) t -> ('b, 'a) t
    ...
  end
```

One can verify that the concrete and abstract encodings respect the specialization subtyping hierachy. However, they also satisfy a symmetry that makes it possible to write single instances of the `zero` and `one` constructor functions that apply to all specializations. In particular, note that the type of the `one` function makes it explicit that the parity of the bit string is flipped.

7 Conclusion

The phantom types technique uses the definition of type equivalence in SML to encode information in a free type variable of a type. Unification can then be used to enforce a particular structure on the information carried by two such types. In this paper, we have focused on encoding subtyping information. We were able to provide encodings for hierarchies with various characteristics, and more generally, hinted at a theory for how such encodings can be derived. Because the technique relies on encoding the subtyping hierarchy, the problem of extensibility arises: how resilient are the encodings to additions to the subtyping hierarchy? This is especially important when designing library interfaces. We showed in this paper that our encodings can handle extensions to the subtyping hierarchy as long as the extensions are always made with respect to a single parent in the hierarchy. We also showed how to extend the techniques we developed to encode a form of prenex bounded polymorphism, with subsumption occuring only at type application. The correctness of this encoding is established by showing how a calculus with that form of subtyping can be translated faithfully (using the encoding) into a calculus embodying the type system of SML.

It goes without saying that this approach to encoding subtyping is not without its problems from a practical point of view. As the encodings in this paper show, the types involved can become quite large. Type abbreviations can help simplify the presentation of concrete types, but for abstract encodings, which require type variables, those type variables need to appear in the interface. Having such complex types lead to interfaces themselves becoming complex, and, more seriously, the type errors reported to the user are fairly unreadable. Although the process of encoding the subtyping hierarchies can be automated, deriving the encodings from a declarative description of the hierarchy, we see no good solution for the complexity problem. The compromise between providing safety and complicating the interface must be decided on a per-case basis.

We demonstrated the utility of the phantom types technique in modern programming languages, like SML or Haskell, by showing that it can be used to capture programming invariants for user-defined datatypes. By modelling an abstract datatype as a collection of constructor, destructor, and coercion functions, we candefine a notion of datatype specialization using the techniques developed in this paper. We further described methods by which the clumsy destructor functions can be replaced by familiar pattern-matching by injecting into and projecting from datatypes inspired by recursion schemes.

We also note that the source language of Section 5 provides only a lower bound on the power of phantom types. For example, one can use features of the *specific* encoding used to further constrain or refine the type of operations. This is used, for instance, in the second `BIT` signature of Section 6.3, and is also used by Reppy [1996] to type socket operations. There is yet no general methodology for exploiting properties of encodings beyond them respecting the subtyping hierarchy.

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A A width-based encoding for finite lattices

We describe in this appendix an encoding for finite lattices which is in general more efficient than the encoding derived from embedding the lattice into a powerset lattice, as defined in Section 3.3.

Let $(L, \leq_L)$ be a lattice we wish to encode. The *width* of L is the maximal size of sets of incomparable elements. Formally,

$$w(L) = \max\{|X| \ : \ X \subseteq L, \forall x, y \in L, (x \not\leq y \ \& \ y \not\leq x)\}.$$

The following proposition allows us to derive an encoding based on the width of the lattice.

Proposition A.1 *Let L be a finite lattice, and w be the width of L. There exists a function $l : L \to \mathbb{N}^w$ such that $x \leq_L y$ iff for $i = 1, \ldots, w$, $l(x)(i) \geq l(y)(i)$.*

Proof: Choose $S = \{s_1, \ldots, s_w\}$ a subset of L such that S is a set of mutually incomparable elements of size w. We iteratively define a function $l' : L \to \mathbb{Q}$. We initially set $l'(\top_L) = (0, \ldots, 0)$, $l'(\bot_L) = (1, \ldots, 1)$, and for every s_i in the set S,

$$l'(s_i) = (a_1, \ldots, a_w) \quad \text{where } a_k = \begin{cases} \frac{1}{2} & \text{if } i \neq k \\ 0 & \text{otherwise} \end{cases}$$

Iteratively, for all elements $x \in L$ not assigned a value by l', define the sets

$$x^{>} = \{y \in L \ : \ y \text{ is assigned a value by } l' \text{ and } y > x\}$$

and

$$x^{<} = \{y \in L \ : \ y \text{ is assigned a value by } l' \text{ and } y < x\}.$$

It is easy to verify that either $x^{>} \cap S \neq \emptyset$ or $x^{<} \cap S \neq \emptyset$, but not both (otherwise, there exists $y^{<} \in S$ and $y^{>} \in S$ such that $y^{>} > x > y^{<}$, and hence $y^{>} > y^{<}$, contradicting the mutual incomparability of elements of S). Define $l'(x) = (x_1, \ldots, x_n)$, where

$$x_i = \frac{\min_{y \in x^{<}}\{l'(y)(i)\} + \max_{y \in x^{>}}\{l'(y)(i)\}}{2}.$$

We can now define the function $l : L \to \mathbb{N}^w$ by simply rescaling the result of the function l'. Let R be the sequence of all the rational numbers that appear in some tuple position in the result $l'(x)$ for some $x \in L$, ordered by the standard order on $\mathbb{Q}$. For $r \in R$, let $i(r)$ be the index of the rational number r in R. Define the function l by $l(x) = (i(l'(x)(1)), \ldots, i(l'(x)(w)))$. It is straightforward to verify that the property in the proposition holds for this function. ∎

We can use Proposition A.1 to encode elements of a finite lattice L. Define a datatype

```
datatype α z = Z
```

(as usual, the data constructor name is irrelevant). We encode an element into a tuple of size w, the width of L. Assume we have a labelling of the elements of L by a function l as given by Proposition A.1. Essentially, l will indicate the nesting of the above type constructor in the encoding. Formally,

$$\begin{aligned} \langle X \rangle_C &= \underbrace{(\ldots(\texttt{unit z})\ldots\texttt{z})}_{l(x)(1)} \times \ldots \times \underbrace{(\ldots(\texttt{unit z})\ldots\texttt{z})}_{l(x)(w)} \\ \langle X \rangle_A &= \underbrace{(\ldots(\alpha_1\ \texttt{z})\ldots\texttt{z})}_{l(x)(1)} \times \ldots \times \underbrace{(\ldots(\alpha_w\ \texttt{z})\ldots\texttt{z})}_{l(x)(w)} \end{aligned}$$

(As usual, each α_i in $\langle \cdot \rangle_A$ is fresh.)

B The calculus $\lambda^{\mathrm{DM}}_{<:}$

Types:

$\tau ::=$	Monotypes
t	Basic type $(t \in T)$
α	Type variable
$\tau_1 \to \tau_2$	Function type
$\sigma ::=$	Prenex quantified type scheme
$\forall \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n . \tau$	

We make a syntactic restriction that precludes the use of type variables in the bounds of quantified type variables.

Expression syntax:

$e ::=$	Monomorphic expressions
c	Constant $(c \in C_b \cup C_p)$
$\lambda x : \tau . e$	Functional abstraction
$e_1 \; e_2$	Function application
x	Variable
$p \; [\tau_1, \ldots, \tau_n]$	Type application
let $x = p$ **in** e	Local binding
$p ::=$	Polymorphic expressions
x	Variable
$\Lambda \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n . e$	Type abstraction
$v ::=$	Values
c	Constant $(c \in C_b \cup C_p)$
$\lambda x : \tau . e$	Functional abstraction

Evaluation contexts:

$E ::=$	Evaluation contexts
$[\,]$	Empty context
$E \; e$	Application context
$v \; E$	Argument context
$E \; [\tau_1, \ldots, \tau_n]$	Type application context

Operational semantics:

$(\lambda x : \tau . e) \; v \longrightarrow_{<:} e\{v/x\}$

$(\Lambda \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n . e) \; [\tau_1', \ldots, \tau_n'] \longrightarrow_{<:} e\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\}$

let $x = p$ **in** $e \longrightarrow_{<:} e\{p/x\}$

$c_1 \; c_2 \longrightarrow_{<:} c_3$ iff $\delta(c_1, c_2) = c_3$

$E[e_1] \longrightarrow_{<:} E[e_2]$ iff $e_1 \longrightarrow_{<:} e_2$

The function $\delta : C_p \times C_b \rightharpoonup C_p$ is a partial function defining the result of applying a primitive operation to a basic constant.

Typing contexts:

$\Gamma ::=$	Type environments
$\quad \cdot$	Empty
$\quad \Gamma, x : \tau$	Monotype
$\quad \Gamma, x : \sigma$	Type scheme
$\Delta ::=$	Subtype environments
$\quad \cdot$	Empty
$\quad \Delta, \alpha <: \tau$	Subtype

Judgments:

$\vdash_{<:} \Delta$ ctxt	Good context Δ
$\Delta \vdash_{<:} \tau$ type	Good monotype τ
$\Delta \vdash_{<:} \sigma$ scheme	Good type scheme σ
$\Delta \vdash_{<:} \Gamma$ ctxt	Good context Γ
$\Delta \vdash_{<:} \tau_1 <: \tau_2$	Type τ_1 subtype of τ_2
$\Delta; \Gamma \vdash_{<:} e : \tau$	Good expression e with monotype τ
$\Delta; \Gamma \vdash_{<:} p : \sigma$	Good expression p with type scheme σ

Judgment $\vdash_{<:} \Delta$ ctxt**:**

$$\frac{}{\vdash_{<:} \cdot \text{ ctxt}} \qquad \frac{\vdash_{<:} \Delta \text{ ctxt} \quad \Delta \vdash_{<:} \tau \text{ type}}{\vdash_{<:} \Delta, \alpha <: \tau \text{ ctxt}}$$

Judgment $\Delta \vdash_{<:} \tau$ type**:**

$$\frac{}{\Delta \vdash_{<:} t \text{ type}} \qquad \frac{\vdash_{<:} \Delta \text{ ctxt} \quad \alpha \in \text{dom}(\Delta)}{\Delta \vdash_{<:} \alpha \text{ type}} \qquad \frac{\Delta \vdash_{<:} \tau_1 \text{ type} \quad \Delta \vdash_{<:} \tau_2 \text{ type}}{\Delta \vdash_{<:} \tau_1 \to \tau_2 \text{ type}}$$

Judgment $\Delta \vdash_{<:} \sigma$ scheme**:**

$$\frac{\Delta, \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n \vdash_{<:} \tau \text{ type}}{\Delta \vdash_{<:} \forall \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n . \tau \text{ scheme}}$$

Judgment $\Delta \vdash_{<:} \Gamma$ ctxt**:**

$$\frac{}{\Delta \vdash_{<:} \cdot \text{ ctxt}} \qquad \frac{\Delta \vdash_{<:} \Gamma \text{ ctxt} \quad \Delta \vdash_{<:} \tau \text{ type}}{\Delta \vdash_{<:} \Gamma, x : \tau \text{ ctxt}} \qquad \frac{\Delta \vdash_{<:} \Gamma \text{ ctxt} \quad \Delta \vdash_{<:} \sigma \text{ scheme}}{\Delta \vdash_{<:} \Gamma, x : \sigma \text{ ctxt}}$$

Judgment $\Delta \vdash_{<:} \tau_1 <: \tau_2$**:**

$$\frac{}{\Delta \vdash_{<:} \tau <: \tau} \qquad \frac{}{\Delta, \alpha <: \tau \vdash_{<:} \alpha <: \tau} \qquad \frac{t_1 \leq t_2}{\Delta \vdash_{<:} t_1 <: t_2}$$

$$\frac{\Delta \vdash_{<:} \tau_2 <: \tau_3}{\Delta \vdash_{<:} \tau_1 \rightarrow \tau_2 <: \tau_1 \rightarrow \tau_3} \qquad \frac{\Delta \vdash_{<:} \tau_1 <: \tau_2 \quad \Delta \vdash_{<:} \tau_2 <: \tau_3}{\Delta \vdash_{<:} \tau_1 <: \tau_3}$$

Judgment $\Delta; \Gamma \vdash_{<:} e : \tau$**:**

$$\frac{}{\Delta; \Gamma \vdash_{<:} c : \pi_b(c)} \; (c \in C_b)$$

$$\frac{\tau\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\} \in \pi_p(c) \quad \forall \tau_1' <: \tau_1, \ldots, \tau_n' <: \tau_n}{\Delta, \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n; \Gamma \vdash_{<:} c : \tau' \rightarrow \tau} \; \left(\begin{array}{c} c \in C_p, \\ FV(\tau') = \langle \alpha_1, \ldots, \alpha_n \rangle \end{array} \right)$$

$$\frac{\Delta \vdash_{<:} \Gamma \text{ ctxt}}{\Delta; \Gamma, x : \tau \vdash_{<:} x : \tau} \qquad \frac{\Delta; \Gamma, x : \tau \vdash_{<:} e : \tau'}{\Delta; \Gamma \vdash_{<:} \lambda x{:}\tau.e : \tau \rightarrow \tau'}$$

$$\frac{\Delta; \Gamma \vdash_{<:} e_1 : \tau_1 \rightarrow \tau_2 \quad \Delta; \Gamma \vdash_{<:} e_2 : \tau_1}{\Delta; \Gamma \vdash_{<:} e_1 \; e_2 : \tau_2}$$

$$\frac{\Delta; \Gamma \vdash_{<:} p : \forall \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau \quad \Delta \vdash_{<:} \tau_1' <: \tau_1 \quad \cdots \quad \Delta \vdash_{<:} \tau_n' <: \tau_n}{\Delta; \Gamma \vdash_{<:} p \; [\tau_1', \ldots, \tau_n'] : \tau\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\}}$$

$$\frac{\Delta; \Gamma, x : \sigma \vdash_{<:} e : \tau \quad \Delta; \Gamma \vdash_{<:} p : \sigma}{\Delta; \Gamma \vdash_{<:} \textbf{let } x = p \textbf{ in } e : \tau}$$

Judgment $\Delta; \Gamma \vdash_{<:} p : \sigma$**:**

$$\frac{\Delta \vdash_{<:} \Gamma \text{ ctxt} \quad \Delta, \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n; \Gamma \vdash_{<:} e : \tau}{\Delta; \Gamma \vdash_{<:} \Lambda \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.e : \forall \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau} \; (\alpha_1, \ldots, \alpha_n \notin \Delta)$$

B.1 Proofs

Lemma B.1 *(Monotype expression substitution preserves typing)*

- *If* $\Delta; \Gamma, x : \tau' \vdash_{<:} e : \tau$ *and* $\Delta; \Gamma \vdash_{<:} e' : \tau'$*, then* $\Delta; \Gamma \vdash_{<:} e\{e'/x\} : \tau$.
- *If* $\Delta; \Gamma, x : \tau' \vdash_{<:} p : \sigma$ *and* $\Delta; \Gamma \vdash_{<:} e' : \tau'$*, then* $\Delta; \Gamma \vdash_{<:} p\{e'/x\} : \tau$.

Proof: Proceed by simultaneous induction on the derivations $\Delta; \Gamma, x : \tau' \vdash_{<:} e : \tau$ and $\Delta; \Gamma, x : \tau' \vdash_{<:} p : \sigma$.

The cases for constants are immediate.

The cases for variables (both monotype and polytype) is immediate.

In the lambda case, $e = \lambda y : \tau_a.e_a$, $\tau = \tau_a \rightarrow \tau_b$, and $\Delta; \Gamma, x : \tau', y : \tau_a \vdash_{<:} e_a : \tau_b$. Assume $y \neq x, y \notin FV(e')$. Note $(\lambda y{:}\tau_a.e_a)\{e'/x\} = \lambda y{:}\tau_a.e_a\{e'/x\}$. Furthermore, $\Delta; \Gamma, y : \tau_a, x : \tau' \vdash_{<:} e_a : \tau_b$. By the induction hypothesis, $\Delta; \Gamma, y : \tau_a \vdash_{<:} e_a\{e'/x\} : \tau_b$. Hence, $\Delta; \Gamma \vdash_{<:} (\lambda y{:}\tau_a.e_a)\{e'/x\} : \tau$.

In the application case, $e = e_1\ e_2$, $\Delta;\Gamma, x : \tau' \vdash_{<:} e_1 : \tau_a \rightarrow \tau$. and $\Delta;\Gamma, x : \tau' \vdash_{<:} e_2 : \tau_a$. Note $(e_1\ e_2)\{e'/x\} = e_1\{e'/x\}\ e_2\{e'/x\}$. By the induction hypothesis, $\Delta;\Gamma \vdash_{<:} e_1\{e'/x\} : \tau_a \rightarrow \tau$ and $\Delta;\Gamma \vdash_{<:} e_2\{e'/x\} : \tau_a$. Hence, $\Delta;\Gamma \vdash_{<:} (e_1\ e_2)\{e'/x\} : \tau$.

In the type application case, $e = p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}]$, $\tau = \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$, $\Delta;\Gamma, x : \tau' \vdash_{<:} p_a : \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta \vdash_{<:} \tau_{b,1} <: \tau_{a,1}$,. . .,$\Delta \vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. Note $(p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}])\{e'/x\} = p_a\{e'/x\}\ [\tau_{b,1}, \ldots, \tau_{b,n}]$. By the induction hypothesis, $\Delta;\Gamma \vdash_{<:} p_a\{e'/x\} : \sigma$. Hence, $\Delta;\Gamma \vdash_{<:} (p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}])\{e'/x\} : \tau$.

In the local binding case, $e =$ **let** $y = p_a$ **in** e_b, $\Delta;\Gamma, x : \tau' \vdash_{<:} p_a : \sigma_a$, $\Delta;\Gamma, x : \tau', y : \sigma_a \vdash_{<:} e_b : \tau$. Assume $y \neq x, y \notin FV(e')$. Note (**let** $y = p_a$ **in** $e_b)\{e'/x\} =$ **let** $y = p_a\{e'/x\}$ **in** $e_b\{e'/x\}$. Furthermore, $\Delta;\Gamma, y : \sigma, x : \tau' \vdash_{<:} e_b : \tau$. By the induction hypothesis, $\Delta;\Gamma \vdash_{<:} p_a\{e'/x\} : \sigma$ and $\Delta;\Gamma \vdash_{<:} e_b\{e'/x\} : \tau$. Hence, $\Delta;\Gamma \vdash_{<:}$ (**let** $y = p_a$ **in** $e_b)\{e'/x\} : \tau$.

In the type abstraction case, $p = \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$, $\sigma = \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma, x : \tau' \vdash_{<:} e_a : \tau_a$. Assume $\alpha_1, \ldots, \alpha_n \notin FV(e')$. Note $(\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a)\{e'/x\} = \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a\{e'/x\}$. By the induction hypothesis, $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma \vdash_{<:} e_a\{e'/x\} : \tau_a$. Hence, $\Delta;\Gamma \vdash_{<:} (\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a)\{e'/x\} : \sigma$. ∎

Lemma B.2 *(Polytype expression substitution preserves typing)*

- *If* $\Delta;\Gamma, x : \sigma' \vdash_{<:} e : \tau$ *and* $\Delta;\Gamma \vdash_{<:} p' : \sigma'$*, then* $\Delta;\Gamma \vdash_{<:} e\{p'/x\} : \tau$.
- *If* $\Delta;\Gamma, x : \sigma' \vdash_{<:} p : \sigma$ *and* $\Delta;\Gamma \vdash_{<:} p' : \sigma'$*, then* $\Delta;\Gamma \vdash_{<:} p\{p'/x\} : \sigma$.

Proof: Proceed by simultaneous induction on the derivations $\Delta;\Gamma, x : \sigma' \vdash_{<:} e : \tau$ and $\Delta;\Gamma, x : \sigma' \vdash_{<:} p : \sigma$.

The cases for constants are immediate.

The cases for variables (both monotype and polytype) is immediate.

In the lambda case, $e = \lambda y : \tau_a.e_a$, $\tau = \tau_a \rightarrow \tau_b$, and $\Delta;\Gamma, x : \sigma', y : \tau_a \vdash_{<:} e_a : \tau_b$. Assume $y \neq x, y \notin FV(e')$. Note $(\lambda y : \tau_a.e_a)\{p'/x\} = \lambda y : \tau_a.e_a\{p'/x\}$. Furthermore, $\Delta;\Gamma, y : \tau_a, x : \sigma' \vdash_{<:} e_a : \tau_b$. By the induction hypothesis, $\Delta;\Gamma, y : \tau_a \vdash_{<:} e_a\{p'/x\} : \tau_b$. Hence, $\Delta;\Gamma \vdash_{<:} (\lambda y : \tau_a.e_a)\{p'/x\} : \tau$.

In the application case, $e = e_1\ e_2$, $\Delta;\Gamma, x : \sigma' \vdash_{<:} e_1 : \tau_a \rightarrow \tau$. and $\Delta;\Gamma, x : \sigma' \vdash_{<:} e_2 : \tau_a$. Note $(e_1\ e_2)\{p'/x\} = e_1\{p'/x\}\ e_2\{p'/x\}$. By the induction hypothesis, $\Delta;\Gamma \vdash_{<:} e_1\{p'/x\} : \tau_a \rightarrow \tau$ and $\Delta;\Gamma \vdash_{<:} e_2\{p'/x\} : \tau_a$. Hence, $\Delta;\Gamma \vdash_{<:} (e_1\ e_2)\{p'/x\} : \tau$.

In the type application case, $e = p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}]$, $\tau = \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$, $\Delta;\Gamma, x : \sigma' \vdash_{<:} p_a : \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta \vdash_{<:} \tau_{b,1} <: \tau_{a,1}$,. . .,$\Delta \vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. Note $(p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}])\{p'/x\} = p_a\{p'/x\}\ [\tau_{b,1}, \ldots, \tau_{b,n}]$. By the induction hypothesis, $\Delta;\Gamma \vdash_{<:} p_a\{p'/x\} : \sigma$. Hence, $\Delta;\Gamma \vdash_{<:} (p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}])\{p'/x\} : \tau$.

In the local binding case, $e =$ **let** $y = p_a$ **in** e_b, $\Delta;\Gamma, x : \sigma' \vdash_{<:} p_a : \sigma_a$, $\Delta;\Gamma, x : \sigma', y : \sigma_a \vdash_{<:} e_b : \tau$. Assume $y \neq x, y \notin FV(e')$. Note (**let** $y = p_a$ **in** $e_b)\{p'/x\} =$ **let** $y = p_a\{p'/x\}$ **in** $e_b\{p'/x\}$. Furthermore, $\Delta;\Gamma, y : \sigma, x : \sigma' \vdash_{<:} e_b : \tau$. By the induction hypothesis, $\Delta;\Gamma \vdash_{<:} p_a\{p'/x\} : \sigma_a$ and $\Delta;\Gamma \vdash_{<:} e_b\{p'/x\} : \tau$. Hence, $\Delta;\Gamma \vdash_{<:}$ (**let** $y = p_a$ **in** $e_b)\{p'/x\} : \tau$.

In the type abstraction case, $p = \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$, $\sigma = \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma, x : \sigma' \vdash_{<:} e_a : \tau_a$. Assume $\alpha_1, \ldots, \alpha_n \notin FV(e')$. Note $(\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a)\{p'/x\} = \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a\{p'/x\}$. By the induction hypothesis, $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma \vdash_{<:} e_a\{p'/x\} : \tau_a$. Hence, $\Delta; \Gamma \vdash_{<:} (\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a)\{p'/x\} : \sigma$. ∎

Lemma B.3 *(Type subsumption preserves subtyping)*

- *If* $\Delta, \alpha <: \tau' \vdash_{<:} \tau_1 <: \tau_2$*, and* $\tau'' <: \tau'$ *then* $\Delta, \alpha <: \tau'' \vdash_{<:} \tau_1 <: \tau_2$.

Proof: Proceed by induction on the derivation $\Delta, \alpha <: \tau' \vdash_{<:} \tau_1 <: \tau_2$. ∎

Lemma B.4 *(Type subsumption preserves typing)*

- *If* $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} e : \tau$ *and* $\tau'' <: \tau'$*, then* $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} e : \tau$.
- *If* $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p : \sigma$ *and* $\tau'' <: \tau'$*, then* $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} p : \sigma$.

Proof: Proceed by simultaneous induction on the derivations $\Delta, \alpha : \tau'; \Gamma \vdash_{<:} e : \tau$ and $\Delta, \alpha : \tau'; \Gamma \vdash_{<:} p : \sigma$.

The case for basic constants is immediate.

In the primitive operation case, $e = c$ ($c \in C_p$), $FV(\tau) = \langle \alpha_1, \ldots, \alpha_n \rangle$, and $\forall \tau_1^* <: \tau_1, \ldots, \tau_n^* <: \tau_n$, we have $\tau\{\tau_1^*/\alpha_1, \ldots, \tau_n^*/\alpha_n\} \in \pi_p(\tau)$. If $\alpha \neq \alpha_i$, the result is immediate. If $\alpha = \alpha_i$, then $\forall \tau_1^* <: \tau_1, \ldots, \tau_i^* <: \tau', \ldots, \tau_n^* <: \tau_n.\tau\{\tau_1^*/\alpha_1, \ldots, \tau_i^*/\alpha, \ldots, \tau_n^*/\alpha_n\} \in \pi_p(\tau)$ and $\Delta \vdash_{<:} \tau'' <: \tau'$ implies $\forall \tau_1^* <: \tau_1, \ldots, \tau_i^* <: \tau'', \ldots, \tau_n^* <: \tau_n.\tau\{\tau_1^*/\alpha_1, \ldots, \tau_i^*/\alpha_i, \ldots, \tau_n^*/\alpha_n\} \in \pi_p(\tau)$.Thus, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} c : \tau$.

The cases for variables (both monotype and polytype) are immediate.

In the lambda case, $e = \lambda x{:}\tau_a.e_a$, $\tau = \tau_a \to \tau_b$, and $\Delta, \alpha <: \tau'; \Gamma, x : \tau_a \vdash_{<:} e_a : \tau_b$. By the induction hypothesis, $\Delta, \alpha <: \tau''; \Gamma, x : \tau_a \vdash_{<:} e_a : \tau_b$. Hence, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} \lambda x{:}\tau_a.e_a : \tau$.

In the application case, $e = e_1\ e_2$, $\Delta, \alpha <: \tau'; \Gamma, x : \sigma' \vdash_{<:} e_1 : \tau_a \to \tau$. and $\Delta, \alpha <: \tau'; \Gamma, x : \sigma' \vdash_{<:} e_2 : \tau_a$. By the induction hypothesis, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} e_1 : \tau_a \to \tau$ and $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} e_2 : \tau_a$. Hence, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} e_1\ e_2 : \tau$.

In the type application case, $e = p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}]$, $\tau = \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$, $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p_a : \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta, \alpha <: \tau' \vdash_{<:} \tau_{b,1} <: \tau_{a,1}$,. . .,$\Delta, \alpha <: \tau' \vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. Assume $\alpha_1, \ldots, \alpha_n \neq \alpha$. By the induction hypothesis, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} p_a : \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$. By Lemma B.3 $\Delta, \alpha <: \tau'' \vdash_{<:} \tau_{b,1} <: \tau_{a,1}$,. . .,$\Delta, \alpha <: \tau'' \vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. Hence, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}] : \tau$.

In the local binding case, $e =$ **let** $x = p_a$ **in** e_b, $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p_a : \sigma_a$, $\Delta, \alpha <: \tau'; \Gamma, x : \sigma_a \vdash_{<:} e_b : \tau$. By the induction hypothesis, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} p_a : \sigma_a$ and $\Delta, \alpha <: \tau''; \Gamma, x : \sigma \vdash_{<:} e_b : \tau$. Hence, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:}$ **let** $x = p_a$ **in** $e_b : \tau$.

In the type abstraction case, $p = \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$, $\sigma = \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta, \alpha <: \tau', \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma \vdash_{<:} e_a : \tau_a$. Assume $\alpha_1, \ldots, \alpha_n \neq \alpha$. Then, $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}, \alpha <: \tau'; \Gamma \vdash_{<:} e_a : \tau_a$. By the induction hypothesis, $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}, \alpha <: \tau''; \Gamma \vdash_{<:} e_a : \tau_a$. Furthermore, $\Delta, \alpha <: \tau'', \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma \vdash_{<:} e_a : \tau_a$. Hence, $\Delta, \alpha <: \tau''; \Gamma \vdash_{<:} \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a : \sigma$. ∎

Lemma B.5 *(Type substitution preserves subtyping)*

- *If* $\Delta, \alpha <: \tau' \vdash_{<:} \tau_1 <: \tau_2$*, then* $\Delta \vdash_{<:} \tau_1\{\tau'/\alpha\} <: \tau_2\{\tau'/\alpha\}$.

Proof: Proceed by induction on the derivation $\Delta, \alpha <: \tau' \vdash_{<:} \tau_1 <: \tau_2$. ∎

Lemma B.6 *(Type substitution preserves typing)*

- *If* $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} e : \tau$ *then* $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} e\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.
- *If* $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p : \sigma$ *then* $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} p\{\tau'/\alpha\} : \sigma\{\tau'/\alpha\}$.

Proof: Proceed by simultaneous induction on the derivations $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} e : \tau$ and $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p : \sigma$.

The case for basic constants is immediate.

In the primitive operation case, $e = c$ ($c \in C_p$), $FV(\tau) = \langle \alpha_1, \ldots, \alpha_n \rangle$, and $\forall \tau_1^* <: \tau_1, \ldots, \tau_n^* <: \tau_n$ we have $\tau\{\tau_1^*/\alpha_1, \ldots, \tau_n^*/\alpha_n\} \in \pi_p(\tau)$. Note $c\{\tau'/\alpha\} = c$. If $\alpha \neq \alpha_i$, the result is immediate. If $\alpha = \alpha_i$, then $\forall \tau_1^* <: \tau_1, \ldots, \tau_i^* <: \tau', \ldots, \tau_n^* <: \tau_n$ we have $\tau\{\tau_1^*/\alpha_1, \ldots, \tau_i^*/\alpha, \ldots, \tau_n^*/\alpha_n\} \in \pi_p(\tau)$, which implies that $\forall \tau_1^* <: \tau_1, \ldots, \tau_{i-1}^* <: \tau_{i-1}, \tau_{i+1}^* <: \tau_{i+1}, \ldots, \tau_n^* <: \tau_n$, we have $\tau\{\tau'/\alpha\}\{\tau_1^*/\alpha_1, \ldots, \tau_{i-1}^*/\alpha_{i-1}, \tau_{i+1}^*/\alpha_{i+1}, \ldots, \tau_n^*/\alpha_n\} \in \pi_p(\tau)$. Thus, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} c : \tau\{\tau'/\alpha\}$.

The cases for variables (both monotype and polytype) are immediate.

In the lambda case, $e = \lambda x : \tau_a.e_a$, $\tau = \tau_a \to \tau_b$, and $\Delta, \alpha <: \tau'; \Gamma, x : \tau_a \vdash_{<:} e_a : \tau_b$. Note $(\lambda x\!:\!\tau_a.e_a)\{\tau'/\alpha\} = \lambda x\!:\!\tau_a\{\tau'/\alpha\}.e_a\{\tau'/\alpha\}$. By the induction hypothesis, $\Delta; \Gamma\{\tau'/\alpha\}, x : \tau_a\{\tau'/\alpha\} \vdash_{<:} e_a\{\tau'/\alpha\} : \tau_b\{\tau'/\alpha\}$. Hence, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} (\lambda x\!:\!\tau_a.e_a)\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the application case, $e = e_1\ e_2$, $\Delta, \alpha <: \tau'; \Gamma, x : \sigma' \vdash_{<:} e_1 : \tau_a \to \tau$. and $\Delta, \alpha <: \tau'; \Gamma, x : \sigma' \vdash_{<:} e_2 : \tau_a$. Note $(e_1\ e_2)\{\tau'/\alpha\} = e_1\{\tau'/\alpha\}\ e_2\{\tau'/\alpha\}$ and $(\tau_a \to \tau)\{\tau'/\alpha\} = \tau_a\{\tau'/\alpha\} \to \tau\{\tau'/\alpha\}$. By the induction hypothesis, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} e_1\{\tau'/\alpha\} : (\tau_a \to \tau)\{\tau'/\alpha\}$ and $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} e_2\{\tau'/\alpha\} : \tau_a\{\tau'/\alpha\}$. Hence, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} (e_1\ e_2)\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the type application case, $e = p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}]$, $\tau = \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$, $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p_a : \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta, \alpha <: \tau' \vdash_{<:} \tau_{b,1} <: \tau_{a,1}$,. . .,$\Delta, \alpha <: \tau' \vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. Assume $\alpha_1, \ldots, \alpha_n \neq \alpha$. Note $(p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}])\{\tau'/\alpha\} = p_a\{\tau'/\alpha\}\ [\tau_{b,1}\{\tau'/\alpha\}, \ldots, \tau_{b,n}\{\tau'/\alpha\}]$. By the induction hypothesis, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} p_a\{\tau'/\alpha\} : (\forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a)\{\tau'/\alpha\}$. By Lemma B.5 $\Delta \vdash_{<:} \tau_{b,1}\{\tau'/\alpha\} <: \tau_{a,1}\{\tau'/\alpha\}$,. . .,$\Delta \vdash_{<:} \tau_{b,n}\{\tau'/\alpha\} <: \tau_{a,n}\{\tau'/\alpha\}$. Hence, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} (p_a\ [\tau_{b,1}, \ldots, \tau_{b,n}])\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the local binding case, $e = \textbf{let}\ x = p_a\ \textbf{in}\ e_b$, $\Delta, \alpha <: \tau'; \Gamma \vdash_{<:} p_a : \sigma_a$, $\Delta, \alpha <: \tau'; \Gamma, x : \sigma_a \vdash_{<:} e_b : \tau$. Note $(\textbf{let}\ x = p_a\ \textbf{in}\ e_b)\{\tau'/\alpha\} = \textbf{let}\ x = p_a\{\tau'/\alpha\}\ \textbf{in}\ e_b\{\tau'/\alpha\}$. By the induction hypothesis, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} p_a\{\tau'/x\} : \sigma_a\{\tau'/\alpha\}$ and $\Delta; \Gamma\{\tau'/\alpha\}, x : \sigma\{\tau'/\alpha\} \vdash_{<:} e_b\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$. Hence, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} (\textbf{let}\ x = p_a\ \textbf{in}\ e_b)\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the type abstraction case, $p = \Lambda \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$, $\sigma = \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\Delta, \alpha <: \tau', \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma \vdash_{<:} e_a : \tau_a$. Assume $\alpha_1, \ldots, \alpha_n \neq \alpha$. Note $(\Lambda \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a)\{\tau'/\alpha\} = \Lambda \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a\{\tau'/\alpha\}$ and $(\forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a)\{\tau'/\alpha\} = \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a\{\tau'/\alpha\}$ (because type variables are precluded from the types of quantified type variables). Futhermore, $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}, \alpha <: \tau'; \Gamma \vdash_{<:} e_a : \tau_a$. By the induction hypothesis, $\Delta, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \Gamma\{\tau'/\alpha\} \vdash_{<:} e_a\{\tau'/\alpha\} : \tau_a\{\tau'/\alpha\}$. Hence, $\Delta; \Gamma\{\tau'/\alpha\} \vdash_{<:} (\Lambda \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a)\{\tau'/\alpha\} : \sigma\{\tau'/\alpha\}$. ∎

Theorem B.7 *(Preservation) If π_p is sound with respect to δ, $\vdash_{<:} e : \tau$ and $e \longrightarrow_{<:} e'$, then $\vdash_{<:} e' : \tau$.*

Proof: Proceed by induction on the derivation $\vdash_{<:} e : \tau$.

The cases for constants and lambda-abstractions are immediate, since there are no evaluation rules for constants or lambda-abstractions.

In the application case, $e = e_1\ e_2$, $\vdash_{<:} e_1 : \tau_a \to \tau$, and $\vdash_{<:} e_2 : \tau_a$. By the definition of the evaluation relation, either $e_1 \longrightarrow_{<:} e_1'$ and $e' = e_1'\ e_2$, or e_1 is a value, $e_e \longrightarrow_{<:} e_2'$ and $e' = e_1\ e_2'$, or $e_1 = \lambda x\!:\!\tau_a.e_1'$, e_2 is a value, and $e' = e_1'\{e_2/x\}$, or $e_1 = c_1$ ($c_1 \in C_p$), $e_2 = c_2$ ($c_2 \in C_b$), and $e' = \delta(c_1, c_2)$. In the first case, the result follows from the induction hypothesis and the typing judgment for applications. Likewise, in the second case, the result follows from the induction hypothesis and the typing judgment for applications. In the third case, $\cdot;\cdot, x : \tau_a \vdash_{<:} e_1' : \tau$ and the result follows from Lemma B.1. In the fourth case, the result follows by the definition of π_p sound with respect to δ.

In the type application case, $e = p\ [\tau_{b,1}, \ldots, \tau_{b,n}]$, $\vdash_{<:} p : \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\vdash_{<:} \tau_{b,1} <: \tau_{a,1}$,. . .,$\vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. Furthermore, $\tau = \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$. By the definition of evaluation relation, $p = \Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$ and $e' = e_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$. We are required to show $\vdash_{<:} e_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\} : \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$. From $\vdash_{<:} p : \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, we have that $\cdot, \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \cdot \vdash_{<:} e_a : \tau_a$. By (repeated applications of) Lemma B.4, $\cdot, \alpha_1 <: \tau_{b,1}, \ldots, \alpha_n <: \tau_{b,n}; \cdot \vdash_{<:} e_a : \tau_a$. By (repeated applications of) Lemma B.6, $\vdash_{<:} e_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\} : \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$.

In the local binding case, $e =$ **let** $x = p_a$ **in** e_b, $\vdash_{<:} p_a : \sigma_a$, and $\cdot;\cdot, x : \sigma_a \vdash_{<:} e_b : \tau$. By the definition of the evaluation relation, $e' = e_b\{p_a/x\}$. The result follows from Lemma B.2. ∎

Lemma B.8 *(Canonical Forms)*

- *If $\vdash_{<:} v : t$, then v has the form c ($c \in C_b$).*
- *If $\vdash_{<:} v : \tau_a \to \tau_b$, then either v has the form c ($c \in C_p$) or v has the form $\lambda x\!:\!\tau_a.e_a$.*
- *If $\vdash_{<:} p : \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, then p has the form $\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$*

Proof: For the first part, proceed by case analysis of the derivation $\vdash_{<:} v : t$. By inspection of the typing rules, it is clear that the final rule in a derivation of $\vdash_{<:} v : t$ must be the rule for basic constants. Hence, v has the form c ($c \in C_b$).

For the second part, proceed by case analysis of the derivation $\vdash_{<:} v : \tau_a \to \tau_b$. By inspection of the typing rules, it is clear that the final rule in a derivation of $\vdash_{<:} v : \tau_a \to \tau_b$ must be the rule for primitive constants or the rule for lambda-abstractions. Hence either v has the form c ($c \in C_p$) or v has the form $\lambda x\!:\!\tau_a.e_a$.

For the third part, proceed by case analysis of p. Suppose p is of the form x. Then $\vdash_{<:} x : \forall\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$ and $\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \cdot \vdash_{<:} x : \tau_a$. But, $x \notin \mathrm{dom}(\cdot)$, hence $\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}; \cdot \vdash_{<:} x : \tau_a$ cannot have a proper derivation. Thus, p is not of the form x. Therefore, p must have the form $\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$. ∎

Theorem B.9 *(Progress) If π_p is sound with respect to δ and $\vdash_{<:} e : \tau$, then either e is a value or there exists e' such that $e \longrightarrow_{<:} e'$.*

Proof: Proceed by induction on the derivation $\vdash_{<:} e : \tau$.

The two cases for constants are immediate, since constants are values. Likewise, the case for lambda-abstraction is immediate, since a lambda-abstraction is a value.

The variable case cannot occur, because the type environment is empty.

In the application case, $e = e_1\ e_2$, $\vdash_{<:} e_1 : \tau_a \rightarrow \tau$. and $\vdash_{<:} e_2 : \tau_a$. By the induction hypothesis, either e_1 is a value or there e_1 can take a step. If e_1 can take a step, then the application context step applies to e. Otherwise, e_1 is a value. By the induction hypothesis, either e_2 is a value or there e_2 can take a step. If e_2 can take a step, then the application argument step applies to e. Otherwise, e_2 is a value. By part 2 of Theorem B.8, either e_1 has the form c_1 ($c_1 \in C_p$) or $\lambda x\!:\!\tau_a.e_a$. If e_1 has the form $\lambda x\!:\!\tau_a.e_a$, then the application step applies to e. If e_1 has the form c_1 ($c_1 \in C_p$), then $\tau_a \rightarrow \tau = t_a \rightarrow t$ for $t_a, t \in T$ and $t_a \rightarrow t \in \pi_p(c_1)$ by the typing judgment for primitive operations. By part 1 of Theorem B.8, e_2 has the form c_2 ($c_2 \in C_b$). Hence, $\vdash_{<:} c_1\ c_2 : \tau$ and $\delta(c_1, c_2)$ is defined by the definition of π_p sound with respect to δ. Thus, the primitive step applies to e.

In the type application case, $e = p\ [\tau_{b,1}, \ldots, \tau_{b,n}]$, $\vdash_{<:} p : \forall \alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.\tau_a$, and $\vdash_{<:} \tau_{b,1} <: \tau_{a,1}$, ..., $\vdash_{<:} \tau_{b,n} <: \tau_{a,n}$. By part 3 of Theorem B.8, p has the form $\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n <: \tau_{a,n}.e_a$, and the type application step applies to e.

In the local binding case, the local binding step applies to e. ∎

Theorem 5.1: *(Soundness) If π_p is sound with respect to δ, $\vdash_{<:} e : \tau$ and $e \longrightarrow^*_{<:} e'$, then $\vdash_{<:} e' : \tau$ and either e' is a value or there exists e'' such that $e' \longrightarrow_{<:} e''$.*

Proof: We assume $\vdash_{<:} e : \tau$ and $e \longrightarrow^*_{<:} e'$. Then $e \longrightarrow^n_{<:} e'$ for some n. Proceed by induction on n. In the base case, the theorem is equivalent to Theorem B.9. In the step case, the inductive hypothesis, Theorem B.7, and Theorem B.9 suffice to prove the theorem. ∎

C The calculus $\lambda^{\mathrm{DM}}_{\mathsf{T}}$

Types:

$\tau ::=$	Monotypes
α	Type variable
$\tau_1 \rightarrow \tau_2$	Function type
$\mathsf{T}\ \tau$	Type constructor T
1	Unit type
$\tau_1 \times \tau_2$	Product type
$\sigma ::=$	Prenex quantified type scheme
$\forall \alpha_1, \ldots, \alpha_n.\tau$	

Expression syntax:

$e ::=$	Monomorphic expressions
c	Constant ($c \in C_b \cup C_p$)
$\lambda x\!:\!\tau.e$	Functional abstraction
$e_1\ e_2$	Function application

x	Variable
$p\ [\tau_1, \ldots, \tau_n]$	Type application
let $x = p$ **in** e	Local binding
$p ::=$	Polymorphic expressions
x	Variable
$\Lambda\alpha_1 \ldots, \alpha_n.e$	Type abstraction
$v ::=$	Values
c	Constant $(c \in C_b \cup C_p)$
$\lambda x\!:\!\tau.e$	Functional abstraction

Evaluation contexts:

$E ::=$	Evaluation contexts
$[\,]$	Empty context
$E\ e$	Application context
$v\ E$	Argument context
$E\ [\tau_1, \ldots, \tau_n]$	Type application context
let $x = E$ **in** e	Local binding context

Operational semantics:

$(\lambda x\!:\!\tau.e)\ v \longrightarrow_{\mathsf{T}} e\{v/x\}$
$(\Lambda\alpha_1, \ldots, \alpha_n.e)\ [\tau_1', \ldots, \tau_n'] \longrightarrow_{\mathsf{T}} e\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\}$
let $x = p$ **in** $e \longrightarrow_{\mathsf{T}} e\{p/x\}$
$c_1\ c_2 \longrightarrow_{\mathsf{T}} c_3$ iff $\delta(c_1, c_2) = c_3$
$E[e_1] \longrightarrow_{\mathsf{T}} E[e_2]$ iff $e_1 \longrightarrow_{\mathsf{T}} e_2$

The function $\delta : C_p \times C_b \rightharpoonup C_p$ is a partial function defining the result of applying a primitive operation to a basic constant.

Typing contexts:

$\Gamma ::=$	Type environments
$\cdot$	Empty
$\Gamma, x : \tau$	Monotype
$\Gamma, x : \sigma$	Type scheme
$\Delta ::=$	Type variable environments
$\cdot$	Empty
Δ, α	Type variable

Judgments:

$\Delta \vdash_{\mathsf{T}} \Gamma$ ctxt	Good context Γ
$\Delta \vdash_{\mathsf{T}} \tau$ type	Good monotype τ
$\Delta \vdash_{\mathsf{T}} \sigma$ scheme	Good type scheme σ

$\Delta; \Gamma \vdash_{\mathsf{T}} e : \tau$	Good expression e with monotype τ
$\Delta; \Gamma \vdash_{\mathsf{T}} p : \sigma$	Good expression p with type scheme σ

Judgment $\Delta \vdash_{\mathsf{T}} \Gamma$ ctxt**:**

$$\frac{}{\Delta \vdash_{\mathsf{T}} \cdot \text{ ctxt}} \qquad \frac{\Delta \vdash_{\mathsf{T}} \Gamma \text{ ctxt}}{\Delta \vdash_{\mathsf{T}} \Gamma, x : \tau \text{ ctxt}} \qquad \frac{\Delta \vdash_{\mathsf{T}} \Gamma \text{ ctxt}}{\Delta \vdash_{\mathsf{T}} \Gamma, x : \sigma \text{ ctxt}}$$

Judgment $\Delta \vdash_{\mathsf{T}} \tau$ type**:**

$$\frac{}{\Delta \vdash_{\mathsf{T}} t \text{ type}} \qquad \frac{\vdash_{\mathsf{T}} \Delta \text{ ctxt} \quad \alpha \in \text{dom}(\Delta)}{\Delta \vdash_{\mathsf{T}} \alpha \text{ type}} \qquad \frac{\Delta \vdash_{\mathsf{T}} \tau_1 \text{ type} \quad \Delta \vdash_{\mathsf{T}} \tau_2 \text{ type}}{\Delta \vdash_{\mathsf{T}} \tau_1 \to \tau_2 \text{ type}}$$

Judgment $\Delta \vdash_{\mathsf{T}} \sigma$ scheme**:**

$$\frac{\Delta, \alpha_1, \ldots, \alpha_n \vdash_{\mathsf{T}} \tau \text{ type}}{\Delta \vdash_{\mathsf{T}} \forall \alpha_1, \ldots, \alpha_n . \tau \text{ scheme}}$$

Judgment $\Delta; \Gamma \vdash_{\mathsf{T}} e : \tau$**:**

$$\frac{}{\Delta; \Gamma \vdash_{\mathsf{T}} c : \pi_b(c)} \; (c \in C_b)$$

$$\frac{\begin{array}{c}\forall \tau' \in \pi_b(C_b) \text{ with } \mathit{unify}(\tau_1, \tau') = \langle (\alpha_1, \tau'_1), \ldots, (\alpha_n, \tau'_n), \ldots \rangle, \\ (\tau_1 \to \tau_2)\{\tau'_1/\alpha_1, \ldots, \tau'_n/\alpha_n\} \in \pi_p(c)\end{array}}{\Delta, \alpha_1, \ldots, \alpha_n; \Gamma \vdash_{\mathsf{T}} c : \tau_1 \to \tau_2} \; \left(\begin{array}{c} c \in C_p, \\ FV(\tau_1) = \langle \alpha_1, \ldots, \alpha_n \rangle \end{array}\right)$$

$$\frac{\Delta \vdash_{\mathsf{T}} \Gamma \text{ ctxt}}{\Delta; \Gamma, x : \tau \vdash_{\mathsf{T}} x : \tau} \qquad \frac{\Delta; \Gamma, x : \tau \vdash_{\mathsf{T}} e : \tau'}{\Delta; \Gamma \vdash_{\mathsf{T}} \lambda x {:} \tau . e : \tau \to \tau'} \qquad \frac{\Delta; \Gamma \vdash_{\mathsf{T}} e_1 : \tau_1 \to \tau_2 \quad \Delta; \Gamma \vdash_{\mathsf{T}} e_2 : \tau_1}{\Delta; \Gamma \vdash_{\mathsf{T}} e_1 \; e_2 : \tau_2}$$

$$\frac{\Delta; \Gamma \vdash_{\mathsf{T}} p : \forall \alpha_1, \ldots, \alpha_n . \tau}{\Delta; \Gamma \vdash_{\mathsf{T}} p \, [\tau_1, \ldots, \tau_n] : \tau\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\}} \qquad \frac{\Delta; \Gamma, x : \sigma \vdash_{\mathsf{T}} e : \tau \quad \Delta; \Gamma \vdash_{\mathsf{T}} p : \sigma}{\Delta; \Gamma \vdash_{\mathsf{T}} \textbf{let } x = p \textbf{ in } e : \tau}$$

Judgment $\Delta; \Gamma \vdash_{\mathsf{T}} p : \sigma$**:**

$$\frac{\Delta \vdash_{\mathsf{T}} \Gamma \text{ ctxt} \quad \Delta, \alpha_1, \ldots, \alpha_n; \Gamma \vdash_{\mathsf{T}} e : \tau}{\Delta; \Gamma \vdash_{\mathsf{T}} \Lambda \alpha_1, \ldots, \alpha_n . e : \forall \alpha_1, \ldots, \alpha_n . \tau} \; (\alpha_1, \ldots, \alpha_n \notin \Delta)$$

C.1 Proofs

Lemma C.1 *(Monotype expression substitution preserves typing)*

- *If* $\Delta; \Gamma, x : \tau' \vdash_{\mathsf{T}} e : \tau$ *and* $\Delta; \Gamma \vdash_{\mathsf{T}} e' : \tau'$*, then* $\Delta; \Gamma \vdash_{\mathsf{T}} e\{e'/x\} : \tau$.
- *If* $\Delta; \Gamma, x : \tau' \vdash_{\mathsf{T}} p : \sigma$ *and* $\Delta; \Gamma \vdash_{\mathsf{T}} e' : \tau'$*, then* $\Delta; \Gamma \vdash_{\mathsf{T}} p\{e'/x\} : \tau$.

Proof: Proceed by simultaneous induction on the derivations $\Delta;\Gamma,x:\tau'\vdash_{\mathsf{T}} e:\tau$ and $\Delta;\Gamma,x:\tau'\vdash_{\mathsf{T}} p:\sigma$.

The cases for constants are immediate.

The cases for variables (both monotype and polytype) is immediate.

In the lambda case, $e=\lambda y:\tau_a.e_a$, $\tau=\tau_a\rightarrow\tau_b$, and $\Delta;\Gamma,x:\tau',y:\tau_a\vdash_{\mathsf{T}} e_a:\tau_b$. Assume $y\neq x, y\notin FV(e')$. Note $(\lambda y\!:\!\tau_a.e_a)\{e'/x\}=\lambda y\!:\!\tau_a.e_a\{e'/x\}$. Furthermore, $\Delta;\Gamma,y:\tau_a,x:\tau'\vdash_{\mathsf{T}} e_a:\tau_b$. By the induction hypothesis, $\Delta;\Gamma,y:\tau_a\vdash_{\mathsf{T}} e_a\{e'/x\}:\tau_b$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(\lambda y\!:\!\tau_a.e_a)\{e'/x\}:\tau$.

In the application case, $e=e_1\ e_2$, $\Delta;\Gamma,x:\tau'\vdash_{\mathsf{T}} e_1:\tau_a\rightarrow\tau$. and $\Delta;\Gamma,x:\tau'\vdash_{\mathsf{T}} e_2:\tau_a$. Note $(e_1\ e_2)\{e'/x\}=e_1\{e'/x\}\ e_2\{e'/x\}$. By the induction hypothesis, $\Delta;\Gamma\vdash_{\mathsf{T}} e_1\{e'/x\}:\tau_a\rightarrow\tau$ and $\Delta;\Gamma\vdash_{\mathsf{T}} e_2\{e'/x\}:\tau_a$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(e_1\ e_2)\{e'/x\}:\tau$.

In the type application case, $e=p_a\ [\tau_1,\ldots,\tau_n]$, $\tau=\tau_a\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\}$, and $\Delta;\Gamma,x:\tau'\vdash_{\mathsf{T}} p_a:\forall\alpha_1,\ldots,\alpha_n.\tau_a$. Note $(p_a\ [\tau_1,\ldots,\tau_n])\{e'/x\}=p_a\{e'/x\}\ [\tau_1,\ldots,\tau_n]$. By the induction hypothesis, $\Delta;\Gamma\vdash_{\mathsf{T}} p_a\{e'/x\}:\sigma$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(p_a\ [\tau_1,\ldots,\tau_n])\{e'/x\}:\tau$.

In the local binding case, $e=$ **let** $y=p_a$ **in** e_b, $\Delta;\Gamma,x:\tau'\vdash_{\mathsf{T}} p_a:\sigma_a$, $\Delta;\Gamma,x:\tau',y:\sigma_a\vdash_{\mathsf{T}} e_b:\tau$. Assume $y\neq x, y\notin FV(e')$. Note (**let** $y=p_a$ **in** e_b)$\{e'/x\}=$ **let** $y=p_a\{e'/x\}$ **in** $e_b\{e'/x\}$. Furthermore, $\Delta;\Gamma,y:\sigma,x:\tau'\vdash_{\mathsf{T}} e_b:\tau$. By the induction hypothesis, $\Delta;\Gamma\vdash_{\mathsf{T}} p_a\{e'/x\}:\sigma$ and $\Delta;\Gamma\vdash_{\mathsf{T}} e_b\{e'/x\}:\tau$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}$ (**let** $y=p_a$ **in** e_b)$\{e'/x\}:\tau$.

In the type abstraction case, $p=\Lambda\alpha_1,\ldots,\alpha_n.e_a$, $\sigma=\forall\alpha_1,\ldots,\alpha_n.\tau_a$, and $\Delta,\alpha_1,\ldots,\alpha_n;\Gamma,x:\tau'\vdash_{\mathsf{T}} e_a:\tau_a$. Assume $\alpha_1,\ldots,\alpha_n\notin FV(e')$. Note $(\Lambda\alpha_1,\ldots,\alpha_n.e_a)\{e'/x\}=\Lambda\alpha_1,\ldots,\alpha_n.e_a\{e'/x\}$. By the induction hypothesis, $\Delta,\alpha_1,\ldots,\alpha_n;\Gamma\vdash_{\mathsf{T}} e_a\{e'/x\}:\tau_a$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(\Lambda\alpha_1,\ldots,\alpha_n.e_a)\{e'/x\}:\sigma$. ∎

Lemma C.2 *(Polytype expression substitution preserves typing)*

- *If* $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} e:\tau$ *and* $\Delta;\Gamma\vdash_{\mathsf{T}} p':\sigma'$, *then* $\Delta;\Gamma\vdash_{\mathsf{T}} e\{p'/x\}:\tau$.
- *If* $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} p:\sigma$ *and* $\Delta;\Gamma\vdash_{\mathsf{T}} p':\sigma'$, *then* $\Delta;\Gamma\vdash_{\mathsf{T}} p\{p'/x\}:\sigma$.

Proof: Proceed by simultaneous induction on the derivations $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} e:\tau$ and $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} p:\sigma$.

The cases for constants are immediate.

The cases for variables (both monotype and polytype) is immediate.

In the lambda case, $e=\lambda y:\tau_a.e_a$, $\tau=\tau_a\rightarrow\tau_b$, and $\Delta;\Gamma,x:\sigma',y:\tau_a\vdash_{\mathsf{T}} e_a:\tau_b$. Assume $y\neq x, y\notin FV(e')$. Note $(\lambda y\!:\!\tau_a.e_a)\{p'/x\}=\lambda y\!:\!\tau_a.e_a\{p'/x\}$. Furthermore, $\Delta;\Gamma,y:\tau_a,x:\sigma'\vdash_{\mathsf{T}} e_a:\tau_b$. By the induction hypothesis, $\Delta;\Gamma,y:\tau_a\vdash_{\mathsf{T}} e_a\{p'/x\}:\tau_b$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(\lambda y\!:\!\tau_a.e_a)\{p'/x\}:\tau$.

In the application case, $e=e_1\ e_2$, $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} e_1:\tau_a\rightarrow\tau$. and $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} e_2:\tau_a$. Note $(e_1\ e_2)\{p'/x\}=e_1\{p'/x\}\ e_2\{p'/x\}$. By the induction hypothesis, $\Delta;\Gamma\vdash_{\mathsf{T}} e_1\{p'/x\}:\tau_a\rightarrow\tau$ and $\Delta;\Gamma\vdash_{\mathsf{T}} e_2\{p'/x\}:\tau_a$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(e_1\ e_2)\{p'/x\}:\tau$.

In the type application case, $e=p_a\ [\tau_1,\ldots,\tau_n]$, $\tau=\tau_a\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\}$, and $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} p_a:\forall\alpha_1,\ldots,\alpha_n.\tau_a$. Note $(p_a\ [\tau_1,\ldots,\tau_n])\{p'/x\}=p_a\{p'/x\}\ [\tau_1,\ldots,\tau_n]$. By the induction hypothesis, $\Delta;\Gamma\vdash_{\mathsf{T}} p_a\{p'/x\}:\sigma$. Hence, $\Delta;\Gamma\vdash_{\mathsf{T}}(p_a\ [\tau_1,\ldots,\tau_n])\{p'/x\}:\tau$.

In the local binding case, $e=$ **let** $y=p_a$ **in** e_b, $\Delta;\Gamma,x:\sigma'\vdash_{\mathsf{T}} p_a:\sigma_a$, $\Delta;\Gamma,x:\sigma',y:\sigma_a\vdash_{\mathsf{T}} e_b:\tau$. Assume $y\neq x, y\notin FV(e')$. Note (**let** $y=p_a$ **in** e_b)$\{p'/x\}=$ **let** $y=$

$p_a\{p'/x\}$ **in** $e_b\{p'/x\}$. Furthermore, $\Delta;\Gamma, y : \sigma, x : \sigma' \vdash_{\mathsf{T}} e_b : \tau$. By the induction hypothesis, $\Delta;\Gamma \vdash_{\mathsf{T}} p_a\{p'/x\} : \sigma_a$ and $\Delta;\Gamma \vdash_{\mathsf{T}} e_b\{p'/x\} : \tau$. Hence, $\Delta;\Gamma \vdash_{\mathsf{T}}$ (**let** $y = p_a$ **in** $e_b)\{p'/x\} : \tau$.

In the type abstraction case, $p = \Lambda\alpha_1, \ldots, \alpha_n.e_a$, $\sigma = \forall\alpha_1, \ldots, \alpha_n.\tau_a$, and $\Delta, \alpha_1, \ldots, \alpha_n; \Gamma, x : \sigma' \vdash_{\mathsf{T}} e_a : \tau_a$. Assume $\alpha_1, \ldots, \alpha_n \notin FV(e')$. Note $(\Lambda\alpha_1, \ldots, \alpha_n.e_a)\{p'/x\} = \Lambda\alpha_1, \ldots, \alpha_n.e_a\{p'/x\}$. By the induction hypothesis, $\Delta, \alpha_1, \ldots, \alpha_n; \Gamma \vdash_{\mathsf{T}} e_a\{p'/x\} : \tau_a$. Hence, $\Delta;\Gamma \vdash_{\mathsf{T}} (\Lambda\alpha_1, \ldots, \alpha_n.e_a)\{p'/x\} : \sigma$. ∎

Lemma C.3 *(Type substitution preserves typing)*

- *If* $\Delta, \alpha;\Gamma \vdash_{\mathsf{T}} e : \tau$ *then* $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} e\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.
- *If* $\Delta, \alpha;\Gamma \vdash_{\mathsf{T}} p : \sigma$ *then* $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} p\{\tau'/\alpha\} : \sigma\{\tau'/\alpha\}$.

Proof: Proceed by simultaneous induction on the derivations $\Delta, \alpha;\Gamma \vdash_{\mathsf{T}} e : \tau$ and $\Delta, \alpha;\Gamma \vdash_{\mathsf{T}} p : \sigma$.

The case for basic constants is immediate.

In the primitive operation case, $e = c$ ($c \in C_p$), $\tau = \tau_1 \to \tau_2$, $FV(\tau_1) = \langle\alpha_1, \ldots, \alpha_n\rangle$, and for all $\tau_* \in \pi_b(C_b)$ such that $\mathit{unify}(\tau_*, \tau_1) = \langle(\alpha_1, \tau_1'), \ldots, (\alpha_n, \tau_n'), \ldots\rangle$, we have $\tau_1 \to \tau_2\{\tau_1'/\alpha_1, \ldots, \tau_n'/\alpha_n\} \in \pi_p(c)$. Note $c\{\tau'/\alpha\} = c$. If $\alpha \neq \alpha_i$ for any i, the result is immediate. If $\alpha = \alpha_i$ for some i (without loss of generality, let $i = 1$), then for any $\tau_* \in \pi_b(C_b)$ such that $\mathit{unify}(\tau_*, \tau_1\{\tau'/\alpha_1\}) = \langle(\alpha_2, \tau_2'), \ldots, (\alpha_n, \tau_n'), \ldots\rangle$, we have $\mathit{unify}(\tau_*, \tau_1) = \langle(\alpha_1, \tau'), (\alpha_2, \tau_2'), \ldots, (\alpha_n, \tau_n'), \ldots\rangle$, so that $(\tau_1 \to \tau_2)\{\tau'/\alpha\}\{\tau_2'/\alpha_2, \ldots, \tau_n'/\alpha_n\} = (\tau_1 \to \tau_2)\{\tau'/\alpha_1, \tau_2'/\alpha_2, \ldots, \tau_n'/\alpha_n\} \in \pi_p(c)$. Thus, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} c : \tau\{\tau'/\alpha\}$.

The cases for variables (both monotype and polytype) are immediate.

In the lambda case, $e = \lambda x : \tau_a.e_a$, $\tau = \tau_a \to \tau_b$, and $\Delta, \alpha;\Gamma, x : \tau_a \vdash_{\mathsf{T}} e_a : \tau_b$. Note $(\lambda x : \tau_a.e_a)\{\tau'/\alpha\} = \lambda x : \tau_a\{\tau'/\alpha\}.e_a\{\tau'/\alpha\}$. By the induction hypothesis, $\Delta;\Gamma\{\tau'/\alpha\}, x : \tau_a\{\tau'/\alpha\} \vdash_{\mathsf{T}} e_a\{\tau'/\alpha\} : \tau_b\{\tau'/\alpha\}$. Hence, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} (\lambda x{:}\tau_a.e_a)\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the application case, $e = e_1\ e_2$, $\Delta, \alpha;\Gamma, x : \sigma' \vdash_{\mathsf{T}} e_1 : \tau_a \to \tau$. and $\Delta, \alpha;\Gamma, x : \sigma' \vdash_{\mathsf{T}} e_2 : \tau_a$. Note $(e_1\ e_2)\{\tau'/\alpha\} = e_1\{\tau'/\alpha\}\ e_2\{\tau'/\alpha\}$ and $(\tau_a \to \tau)\{\tau'/\alpha\} = \tau_a\{\tau'/\alpha\} \to \tau\{\tau'/\alpha\}$. By the induction hypothesis, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} e_1\{\tau'/\alpha\} : (\tau_a \to \tau)\{\tau'/\alpha\}$ and $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} e_2\{\tau'/\alpha\} : \tau_a\{\tau'/\alpha\}$. Hence, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} (e_1\ e_2)\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the type application case, $e = p_a\ [\tau_1, \ldots, \tau_n]$, $\tau = \tau_a\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\}$, and $\Delta, \alpha;\Gamma \vdash_{\mathsf{T}} p_a : \forall\alpha_1, \ldots, \alpha_n.\tau_a$. Assume $\alpha_1, \ldots, \alpha_n \neq \alpha$. Note $(p_a\ [\tau_1, \ldots, \tau_n])\{\tau'/\alpha\} = p_a\{\tau'/\alpha\}\ [\tau_1\{\tau'/\alpha\}, \ldots, \tau_n\{\tau'/\alpha\}]$. By the induction hypothesis, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} p_a\{\tau'/\alpha\} : (\forall\alpha_1, \ldots, \alpha_n.\tau_a)\{\tau'/\alpha\}$. Hence, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} (p_a\ [\tau_1, \ldots, \tau_n])\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the local binding case, $e =$ **let** $x = p_a$ **in** e_b, $\Delta, \alpha;\Gamma \vdash_{\mathsf{T}} p_a : \sigma_a$, $\Delta, \alpha;\Gamma, x : \sigma_a \vdash_{\mathsf{T}} e_b : \tau$. Note (**let** $x = p_a$ **in** $e_b)\{\tau'/\alpha\} =$ **let** $x = p_a\{\tau'/\alpha\}$ **in** $e_b\{\tau'/\alpha\}$. By the induction hypothesis, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} p_a\{\tau'/x\} : \sigma_a\{\tau'/\alpha\}$ and $\Delta;\Gamma\{\tau'/\alpha\}, x : \sigma\{\tau'/\alpha\} \vdash_{\mathsf{T}} e_b\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$. Hence, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}}$ (**let** $x = p_a$ **in** $e_b)\{\tau'/\alpha\} : \tau\{\tau'/\alpha\}$.

In the type abstraction case, $p = \Lambda\alpha_1, \ldots, \alpha_n.e_a$, $\sigma = \forall\alpha_1, \ldots, \alpha_n.\tau_a$, and $\Delta, \alpha, \alpha_1, \ldots, \alpha_n; \Gamma \vdash_{\mathsf{T}} e_a : \tau_a$. Assume $\alpha_1, \ldots, \alpha_n \neq \alpha$. Note $(\Lambda\alpha_1, \ldots, \alpha_n.e_a)\{\tau'/\alpha\} = \Lambda\alpha_1, \ldots, \alpha_n.e_a\{\tau'/\alpha\}$ and $(\forall\alpha_1, \ldots, \alpha_n.\tau_a)\{\tau'/\alpha\} = \forall\alpha_1, \ldots, \alpha_n.\tau_a\{\tau'/\alpha\}$ (because type variables are precluded from the types of quantified type variables). Futhermore, $\Delta, \alpha_1, \ldots, \alpha_n, \alpha;\Gamma \vdash_{\mathsf{T}} e_a : \tau_a$. By the induction hypothesis, $\Delta, \alpha_1, \ldots, \alpha_n;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} e_a\{\tau'/\alpha\} : \tau_a\{\tau'/\alpha\}$. Hence, $\Delta;\Gamma\{\tau'/\alpha\} \vdash_{\mathsf{T}} (\Lambda\alpha_1 <: \tau_{a,1}, \ldots, \alpha_n : \tau_{a,n}.e_a)\{\tau'/\alpha\} : \sigma\{\tau'/\alpha\}$. ∎

Theorem C.4 *(Preservation) If π_p is sound with respect to δ, $\vdash_{\mathsf{T}} e : \tau$ and $e \longrightarrow_{\mathsf{T}} e'$, then $\vdash_{\mathsf{T}} e' : \tau$.*

Proof: Proceed by induction on the derivation $\vdash_{\mathsf{T}} e : \tau$.

The cases for constants and lambda-abstractions are immediate, since there are no evaluation rules for constants or lambda-abstractions.

In the application case, $e = e_1\ e_2$, $\vdash_{\mathsf{T}} e_1 : \tau_a \to \tau$, and $\vdash_{\mathsf{T}} e_2 : \tau_a$. By the definition of the evaluation relation, either $e_1 \longrightarrow_{\mathsf{T}} e_1'$ and $e' = e_1'\ e_2$, or e_1 is a value, $e_e \longrightarrow_{\mathsf{T}} e_2'$ and $e' = e_1\ e_2'$, or $e_1 = \lambda x : \tau_a . e_1'$, e_2 is a value, and $e' = e_1'\{e_2/x\}$, or $e_1 = c_1$ ($c_1 \in C_p$), $e_2 = c_2$ ($c_2 \in C_b$), and $e' = \delta(c_1, c_2)$. In the first case, the result follows from the induction hypothesis and the typing judgment for applications. Likewise, in the second case, the result follows from the induction hypothesis and the typing judgment for applications. In the third case, $\cdot;\cdot, x : \tau_a \vdash_{\mathsf{T}} e_1' : \tau$ and the result follows from Lemma C.1. In the fourth case, the result follows by the definition of π_p sound with respect to δ.

In the type application case, $e = p\ [\tau_1, \ldots, \tau_n]$, and $\vdash_{\mathsf{T}} p : \forall \alpha_1, \ldots, \alpha_n . \tau_a$. Furthermore, $\tau = \tau_a\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\}$. By the definition of evaluation relation, $p = \Lambda \alpha_1, \ldots, \alpha_n . e_a$ and $e' = e_a\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\}$. We are required to show $\vdash_{\mathsf{T}} e_a\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\} : \tau_a\{\tau_{b,1}/\alpha_1, \ldots, \tau_{b,n}/\alpha_n\}$. From $\vdash_{\mathsf{T}} p : \forall \alpha_1, \ldots, \alpha_n . \tau_a$, we have that $\cdot, \alpha_1, \ldots, \alpha_n; \cdot \vdash_{\mathsf{T}} e_a : \tau_a$. By (repeated applications of) Lemma C.3, $\vdash_{\mathsf{T}} e_a\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\} : \tau_a\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\}$.

In the local binding case, $e = \textbf{let}\ x = p_a\ \textbf{in}\ e_b$, $\vdash_{\mathsf{T}} p_a : \sigma_a$, and $\cdot;\cdot, x : \sigma_a \vdash_{\mathsf{T}} e_b : \tau$. By the definition of the evaluation relation, $e' = e_b\{p_a/x\}$. The result follows from Lemma C.2. ∎

Lemma C.5 *(Canonical Forms)*

- *If $\vdash_{\mathsf{T}} v : \mathsf{T}\ \tau$, then v has the form c ($c \in C_b$).*
- *If $\vdash_{\mathsf{T}} v : \tau_a \to \tau_b$, then either v has the form c ($c \in C_p$) or v has the form $\lambda x : \tau_a . e_a$.*
- *If $\vdash_{\mathsf{T}} p : \forall \alpha_1, \ldots, \alpha_n . \tau_a$, then p has the form $\Lambda \alpha_1, \ldots, \alpha_n . e_a$*

Proof: For the first part, proceed by case analysis of the derivation $\vdash_{\mathsf{T}} v : \mathsf{T}\ \tau$. By inspection of the typing rules, it is clear that the final rule in a derivation of $\vdash_{\mathsf{T}} v : \mathsf{T}\ \tau$ must be the rule for basic constants. Hence, v has the form c ($c \in C_b$).

For the second part, proceed by case analysis of the derivation $\vdash_{\mathsf{T}} v : \tau_a \to \tau_b$. By inspection of the typing rules, it is clear that the final rule in a derivation of $\vdash_{\mathsf{T}} v : \tau_a \to \tau_b$ must be the rule for primitive constants or the rule for lambda-abstractions. Hence either v has the form c ($c \in C_p$) or v has the form $\lambda x : \tau_a . e_a$.

For the third part, proceed by case analysis of p. Suppose p is of the form x. Then $\vdash_{\mathsf{T}} x : \forall \alpha_1, \ldots, \alpha_n . \tau_a$ and $\alpha_1, \ldots, \alpha_n; \cdot \vdash_{\mathsf{T}} x : \tau_a$. But, $x \notin \mathrm{dom}(\cdot)$, hence $\alpha_1, \ldots, \alpha_n; \cdot \vdash_{\mathsf{T}} x : \tau_a$ cannot have a proper derivation. Thus, p is not of the form x. Therefore, p must have the form $\Lambda \alpha_1, \ldots, \alpha_n . e_a$. ∎

Theorem C.6 *(Progress) If π_p is sound with respect to δ and $\vdash_{\mathsf{T}} e : \tau$, then either e is a value or there exists e' such that $e \longrightarrow_{\mathsf{T}} e'$.*

Proof: Proceed by induction on the derivation $\vdash_{\mathsf{T}} e : \tau$.

The two cases for constants are immediate, since constants are values. Likewise, the case for lambda-abstraction is immediate, since a lambda-abstraction is a value.

The variable case cannot occur, because the type environment is empty.

In the application case, $e = e_1\ e_2$, $\vdash_{\mathsf{T}} e_1 : \tau_a \to \tau$. and $\vdash_{\mathsf{T}} e_2 : \tau_a$. By the induction hypothesis, either e_1 is a value or there e_1 can take a step. If e_1 can take a step, then the application context step applies to e. Otherwise, e_1 is a value. By the induction hypothesis, either e_2 is a value or there e_2 can take a step. If e_2 can take a step, then the application argument step applies to e. Otherwise, e_2 is a value. By part 3 of Lemma C.5, either e_1 has the form c_1 ($c_1 \in C_p$) or $\lambda x : \tau_a.e_a$. If e_1 has the form $\lambda x : \tau_a.e_a$, then the application step applies to e. If e_1 has the form c_1 ($c_1 \in C_p$), then $\tau_a \to \tau = t_a \to t$ for $t_a, t \in T$ and $t_a \to t \in \pi_p(c_1)$ by the typing judgment for primitive operations. By part 1 of Theorem C.5, e_2 has the form c_2 ($c_2 \in C_b$). Hence, $\vdash_{\mathsf{T}} c_1\ c_2 : \tau$ and $\delta(c_1, c_2)$ is defined by the definition of π_p sound with respect to δ. Thus, the primitive step applies to e.

In the type application case, $e = p\ [\tau_1, \ldots, \tau_n]$, and $\vdash_{\mathsf{T}} p : \forall\alpha_1, \ldots, \alpha_n.\tau_a$. By part 4 of Lemma C.5, p has the form $\Lambda\alpha_1, \ldots, \alpha_n.e_a$, and the type application step applies to e.

In the local binding case, the local binding step applies to e. ∎

Theorem 5.2: *(Soundness) If π_p is sound with respect to δ, $\vdash_{\mathsf{T}} e : \tau$ and $e \longrightarrow^*_{\mathsf{T}} e'$, then $\vdash_{\mathsf{T}} e' : \tau$ and either e' is a value or there exists e'' such that $e' \longrightarrow_{\mathsf{T}} e''$.*

Proof: We assume $\vdash_{\mathsf{T}} e : \tau$ and $e \longrightarrow^*_{\mathsf{T}} e'$. Then $e \longrightarrow^n_{\mathsf{T}} e'$ for some n. Proceed by induction on n. In the base case, the theorem is equivalent to Theorem B.9. In the step case, the inductive hypothesis, Theorem B.7, and Theorem B.9 suffice to prove the theorem. ∎

D Translation proofs

Theorem 5.3: *If π_p is sound with respect to δ in $\lambda^{\mathrm{DM}}_{<:}$, then $\mathcal{T}[\![\pi_p]\!]$ is sound with respect to δ in $\lambda^{\mathrm{DM}}_{\mathsf{T}}$.*

Proof: We need to show that for all $c_1 \in C_p$ and $c_2 \in C_b$ such that $\vdash_{\mathsf{T}} c_1\ c_2 : \tau$ for some τ, then $\delta(c_1, c_2)$ is defined, and that $\mathcal{T}[\![\pi_b]\!](\delta(c_1, c_2)) = \tau$. Given $c_1 \in C_p$ and $c_2 \in C_b$, assume that $\vdash_{\mathsf{T}} c_1\ c_2 : \tau$. This means that $\vdash_{\mathsf{T}} c_1 : \tau' \to \tau$ and that $\vdash_{\mathsf{T}} c_2 : \tau'$. From $\vdash_{\mathsf{T}} c_1 : \tau' \to \tau$, we derive that for all $\tau^* \in \mathcal{T}[\![\pi_b]\!](C_b)$ such that $\mathit{unify}(\tau^*, \tau') \neq \emptyset$ (since τ' and τ^* are both closed types), $\tau' \to \tau \in \mathcal{T}[\![\pi_p]\!](c_1)$. By definition of $\mathcal{T}$, and by assumption on the form of π_p, this means that $\tau' \to \tau$ is of the form $\mathcal{T}[\![t']\!] \to \mathcal{T}[\![t]\!]$, with $\mathcal{T}[\![t']\!] = \tau'$ and $\mathcal{T}[\![t]\!] = \tau$. Hence, $\vdash_{<:} c_1 : t' \to t$. From $\vdash_{\mathsf{T}} c_2 : \tau'$, we derive that $\mathcal{T}[\![t']\!] = \tau' = \mathcal{T}[\![\pi_b]\!](c_2) = \mathcal{T}[\![\pi_b(c_2)]\!]$. Hence, $\pi_b(c_2) = t'$, and $\vdash_{<:} c_2 : t'$. We can therefore infer that $\vdash_{<:} c_1\ c_2 : t$. Therefore, by soundness of π_p with respect to δ in $\lambda^{\mathrm{DM}}_{<:}$, we get that $\delta(c_1, c_2)$ is defined, and that $\pi_b(\delta(c_1, c_2)) = t$. Thus, $\mathcal{T}[\![\pi_b]\!](\delta(c_1, c_2)) = \mathcal{T}[\![\pi_b(\delta(c_1, c_2))]\!] = \mathcal{T}[\![t]\!] = \tau$, as required. ∎

The following lemma, relating the correcteness of the subtype encoding and substitution, is used in the proof of Theorem 5.4.

Lemma D.1 *For all t,t', and τ with $FV(\tau) \subseteq \langle\alpha\rangle$, if $t^A = \langle t\rangle_A$, $FV(t^A) = \langle\alpha_1, \ldots, \alpha_n\rangle$, and $\mathit{unify}(\langle t'\rangle_C, t^A) = \langle(\alpha_1, \tau_1), \ldots, (\alpha_n, \tau_n), \ldots\rangle$, then $\mathcal{T}[\![\tau\{t'/\alpha\}]\!] = \mathcal{T}[\![\tau]\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1, \ldots, \tau_n/\alpha_n\}$.*

Proof: We proceed by induction on the structure of τ.

For $\tau = \alpha$, we immediately get that $\mathcal{T}[\![\tau\{t'/\alpha\}]\!] = \mathcal{T}[\![t']\!] = \langle t'\rangle_C$. Moreover, we have $\mathcal{T}[\![\tau']\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} = t^A\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} = \langle t'\rangle_C$, by the assumption on the unification of $\langle t'\rangle_C$ and t^A.

For $\tau = t^*$ for some t^*, then $\mathcal{T}[\![\tau\{t'/\alpha\}]\!] = \mathcal{T}[\![\tau]\!] = \langle t^*\rangle_C$. Moreover, $\mathcal{T}[\![\tau]\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} = \mathcal{T}[\![t^*]\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} = \langle t^*\rangle_C\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} = \langle t^*\rangle_C$.

Finally, for $\tau = \tau' \to \tau''$, we have $\mathcal{T}[\![(\tau' \to \tau'')\{t'/\alpha\}]\!] = \mathcal{T}[\![\tau'\{t'/\alpha\} \to \tau''\{t'/\alpha\}]\!] = \mathcal{T}[\![\tau'\{t'/\alpha\}]\!] \to \mathcal{T}[\![\tau''\{t'/\alpha\}]\!]$. By applying the induction hypothesis, this is equal to $\mathcal{T}[\![\tau']\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} \to \mathcal{T}[\![\tau'']\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} = \mathcal{T}[\![\tau' \to \tau'']\!][\alpha \mapsto t^A]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\}$, as required. ∎

Theorem 5.4: *If* $\vdash_{<:} e : \tau$, *then* $\vdash_{\mathsf{T}} \mathcal{E}[\![\vdash_{<:} e : \tau]\!] : \mathcal{T}[\![\tau]\!]$.

Proof: We prove a more general form of this theorem, namely that if $\Delta;\Gamma \vdash_{<:} e : \tau$, then $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![\Delta;\Gamma \vdash_{<:} e : \tau]\!]\rho_\Delta : \mathcal{T}[\![\tau]\!]\rho_\Delta$, where:

$$\mathcal{T}[\![\alpha_1 <: \tau_1,\ldots,\alpha_n <: \tau_n]\!] \triangleq \alpha_{11},\ldots,\alpha_{1k_1},\ldots,\alpha_{n1},\ldots,\alpha_{nk_n} \quad \text{where } \tau_i^A = \mathcal{A}[\![\tau_i]\!] \text{ and } FV(\tau_i^A) = \langle \alpha_{i1},\ldots,\alpha_{ik_i}\rangle$$

and for Δ of the form $\alpha_1 <: \tau_1,\ldots,\alpha_n <: \tau_n$,

$$\rho_\Delta \triangleq \{\alpha_1 \mapsto \tau_1^A,\ldots,\alpha_n \mapsto \tau_n^A\}.$$

Similarly, we show that if $\Delta;\Gamma \vdash_{<:} p : \sigma$, then $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![\Delta;\Gamma \vdash_{<:} p : \sigma]\!]\rho_\Delta : \mathcal{T}[\![\sigma]\!]\rho_\Delta$. We establish this by simultaneous induction on the derivations $\Delta;\Gamma \vdash_{<:} e : \tau$ and $\Delta;\Gamma \vdash_{<:} p : \sigma$.

For variables, $\Delta,\Gamma \vdash_{<:} x : \tau$ implies that $x : \tau$ is in Γ. Hence, $x : \mathcal{T}[\![\tau]\!]\rho_\Delta$ is in $\mathcal{T}[\![\Gamma]\!]\rho_\Delta$. Hence, $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} x : \mathcal{T}[\![\tau]\!]\rho_\Delta$. Similarly for $\Delta;\Gamma \vdash_{<:} x : \sigma$.

For constants $c \in C_b$, if $\Delta;\Gamma \vdash_{<:} c : \tau$, then we have $\pi_b(c) = \tau$. Hence, $\mathcal{T}[\![\pi_b(c)]\!]\rho_\Delta = \mathcal{T}[\![\tau]\!]\rho_\Delta$, and by definition, $\mathcal{T}[\![\pi_b]\!]\rho_\Delta(c) = \mathcal{T}[\![\tau]\!]\rho_\Delta$. This implies $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!] \vdash_{\mathsf{T}} c : \mathcal{T}[\![\tau]\!]\rho_\Delta$.

For constants $c \in C_p$, if $\Delta,\alpha_1 <: \tau_1,\ldots,\alpha_n <: \tau_n;\Gamma \vdash_{<:} c : \tau' \to \tau$ (where $FV(\tau') = \langle\alpha_1,\ldots,\alpha_n\rangle$). Hence, for all $\tau_i' <: \tau_i$, we have $(\tau' \to \tau)\{\tau_1'/\alpha_1,\ldots,\tau_n'/\alpha_n\} \in \pi_p(c)$. Note that this implies that each τ_i' is of the form t_i' for some t_i', due to the restrictions imposed on π_p. Furthermore, also due to the restrictions imposed on π_p, we must have that τ' is either t' for some t', or a type variable α_1. We need to show that for all $\tau^* \in \mathcal{T}[\![\pi_b]\!](C_b)$, if $\mathit{unify}(\tau^*,\mathcal{T}[\![\tau']\!]) = \langle(\alpha_1,\tau_1),\ldots,(\alpha_n,\tau_n),\ldots\rangle$, then $\mathcal{T}[\![\tau' \to \tau]\!]\{\tau_1/\alpha_1,\ldots,\tau_n/\alpha_n\} \in \mathcal{T}[\![\pi_p]\!](c)$. Take an arbitrary $\tau^* \in \mathcal{T}[\![\pi_b]\!](C_b)$. By restrictions on π_b, τ^* is of the form $\langle t^*\rangle_C$ for some $t^* \in \pi_b(C_b)$. Now, consider the different forms of τ'. In the case $\tau' = t'$, we have $\mathcal{T}[\![\tau']\!] = \langle t'\rangle_C$, so that if $\mathit{unify}(\langle t^*\rangle_C,\langle t'\rangle_C)$, then $t^* = t'$. Moreover, because we assumed that concrete encodings did not introduce free type variables, then $FV(\tau') = \emptyset$. Thus, $\mathcal{T}[\![\tau' \to \tau]\!] = \mathcal{T}[\![\tau']\!] \to \mathcal{T}[\![\tau]\!] \in \mathcal{T}[\![\pi_p]\!](c)$ follows immediately from the fact that $\tau' \to \tau \in \pi_p(c)$. In the case that $\tau' = \alpha_1$, then $\mathcal{T}[\![\tau']\!]\rho = \langle t_1'\rangle_A$. Let $FV(t_1') = \langle\alpha_{11},\ldots,\alpha_{1k_1}\rangle$. Assume $\mathit{unify}(\langle t^*\rangle_C,t'^A) = \langle(\alpha_{11},\tau_1),\ldots,(\alpha_{1k_1},\tau_{k_1}),\ldots\rangle$. Because the encoding is respectful, $\mathit{unify}(\langle t^*\rangle_C,t'^A) \neq \emptyset$ if and only if $t^* \leq t_1$, that is, $t^* <: t_1$. By assumption, we have $(\tau' \to \tau)\{t^*/\alpha_1\} \in \pi_p(c)$. Therefore, $\mathcal{T}[\![(\tau' \to \tau)\{t^*/\alpha_1\}]\!] \in \mathcal{T}[\![\pi_p]\!](c)$. By Lemma D.1, $\mathcal{T}[\![(\tau' \to \tau)\{t^*/\alpha_1\}]\!] = \mathcal{T}[\![\tau' \to \tau]\!]\{\tau_1/\alpha_{11},\ldots,\tau_{k_1}/\alpha_{1k_1}\}$, and the result follows. Since τ^* was arbitrary, we can therefore infer that $\mathcal{T}[\![\Delta]\!],\alpha_{11},\ldots,\alpha_{1k_1},\ldots,\alpha_{n1},\ldots,\alpha_{nk_n};\mathcal{T}[\![\Gamma]\!]\rho_\Delta[\alpha_i \mapsto \tau_i^A] \vdash_{\mathsf{T}} c : \mathcal{T}[\![\tau' \to \tau]\!]\rho_\Delta[\alpha_i \mapsto \tau_i^A]$.

For abstractions, if $\Delta;\Gamma \vdash_{<:} \lambda x:\tau'.e : \tau' \rightarrow \tau$, then $\Delta;\Gamma, x : \tau' \vdash_{<:} e : \tau$. By the induction hypothesis, $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta; x : \mathcal{T}[\![\tau']\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![e]\!]\rho_\Delta : \mathcal{T}[\![\tau]\!]\rho_\Delta$, from which one can infer that $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \lambda x:\mathcal{T}[\![\tau']\!]\rho_\Delta.\mathcal{E}[\![e]\!]\rho_\Delta : \mathcal{T}[\![\tau']\!]\rho_\Delta \rightarrow \mathcal{T}[\![\tau]\!]\rho_\Delta$, which yields $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \lambda x:\mathcal{T}[\![\tau']\!]\rho_\Delta.\mathcal{E}[\![e]\!]\rho_\Delta : \mathcal{T}[\![\tau' \rightarrow \tau]\!]\rho_\Delta$.

For applications, if $\Delta;\Gamma \vdash_{<:} e_1\ e_2 : \tau$, then for some τ', $\Delta;\Gamma \vdash_{<:} e_1 : \tau' \rightarrow \tau$ and $\Delta;\Gamma \vdash_{<:} e_2 : \tau'$. By the induction hypothesis, we have $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![e_1]\!] : \mathcal{T}[\![\tau' \rightarrow \tau]\!]$, so that $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![e_1]\!] : \mathcal{T}[\![\tau']\!] \rightarrow \mathcal{T}[\![\tau]\!]$, and $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![e_2]\!] : \mathcal{T}[\![\tau']\!]$. This yields that $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} (\mathcal{E}[\![e_1]\!]\rho_\Delta)\ \mathcal{E}[\![e_2]\!]\rho_\Delta : \mathcal{T}[\![\tau]\!]\rho_\Delta$.

For local bindings, if $\Delta;\Gamma \vdash_{<:} \textbf{let}\ x = p\ \textbf{in}\ e : \tau$, then for some σ we have $\Delta;\Gamma, x : \sigma \vdash_{<:} e : \tau$ and $\Delta;\Gamma \vdash_{<:} p : \sigma$. By the induction hypothesis, we have $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta, x : \mathcal{T}[\![\sigma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![e]\!] : \mathcal{T}[\![\tau]\!]\rho_\Delta$ and $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![p]\!]\rho_\Delta : \mathcal{T}[\![\sigma]\!]\rho_\Delta$. Thus, we have $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \textbf{let}\ x = \mathcal{E}[\![p]\!]\rho_\Delta\ \textbf{in}\ \mathcal{E}[\![e]\!]\rho_\Delta : \mathcal{T}[\![\tau]\!]\rho_\Delta$.

For type applications, we have $\Delta;\Gamma \vdash_{<:} p\ [\tau'_1, \ldots, \tau'_n] : \tau$. Then for all (α_i, τ_i) in $\mathcal{B}[\![p]\!]\Gamma$, we have $\Delta;\Gamma \vdash_{<:} p : \forall\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau'$, $\Delta \vdash_{<:} \tau'_i <: \tau_i$ for all i, and $\tau = \tau'\{\tau'_1/\alpha_1, \ldots, \tau'_n/\alpha_n\}$. By the induction hypothesis, $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![p]\!]\rho_\Delta : \mathcal{T}[\![\forall\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau']\!]\rho_\Delta$. Let $\tau_i^A = \mathcal{A}[\![\tau_i]\!]$, and $FV(\tau_i^A) = \langle\alpha_{i1}, \ldots, \alpha_{ik_i}\rangle$. Thus, $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![p]\!]\rho_\Delta : \forall\alpha_{11}, \ldots, \alpha_{1k_1}, \ldots, \alpha_{n1}, \ldots, \alpha_{nk_n}.\mathcal{T}[\![\tau']\!]\rho_\Delta[\alpha_i \mapsto \tau_i^A]$. We know that $\Delta \vdash_{<:} \tau'_i <: \tau_i$ for all i, so we have that $\mathit{unify}(\tau_i^A, \mathcal{T}[\![\tau_i]\!]\rho_\Delta) = \langle(\alpha_{i1}, \tau_{i1}), \ldots, (\alpha_{ik_i}, \tau_{ik_i}), \ldots\rangle$, and $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![p]\!]\rho_\Delta\ [\tau_{11}, \ldots, \tau_{1k_1}, \ldots, \tau_{n1}, \ldots, \tau_{nk_n}] : \mathcal{T}[\![\tau']\!]\rho_\Delta[\alpha_i \mapsto \tau_i^A]$, as required.

For type abstractions, we have $\Delta;\Gamma \vdash_{<:} \Lambda\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.e : \forall\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau$. Thus, we have $\Delta \vdash_{<:} \Gamma$ ctxt, that is, the type variables in Γ appear in Δ, and moreover $\Delta, \alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n;\Gamma \vdash_{<:} e : \tau$. Let $\tau_i^A = \mathcal{A}[\![\tau_i]\!]$ and $FV(\tau_i^A) = \langle\alpha_{i1}, \ldots, \alpha_{ik_i}\rangle$, for $1 \leq i \leq n$. Let $\rho'_\Delta = \rho_\Delta[\alpha_1 \mapsto \tau_1^A, \ldots, \alpha_n \mapsto \tau_n^A]$. By the induction hypothesis, we have $\mathcal{T}[\![\Delta]\!], \alpha_{11}, \ldots, \alpha_{1k_1}, \ldots, \alpha_{n1}, \ldots, \alpha_{nk_n};\mathcal{T}[\![\Gamma]\!]\rho'_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![e]\!]\rho'_\Delta : \mathcal{T}[\![\tau]\!]\rho'_\Delta$. From this we can infer that $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho'_\Delta \vdash_{\mathsf{T}} \Lambda\alpha_{11}, \ldots, \alpha_{1k_1}, \ldots, \alpha_{n1}, \ldots, \alpha_{nk_n}.\mathcal{E}[\![e]\!]\rho'_\Delta : \forall\alpha_{11}, \ldots, \alpha_{1k_1}, \ldots, \alpha_{n1}, \ldots, \alpha_{nk_n}.\mathcal{T}[\![\tau]\!]\rho'_\Delta$, which is easily seen equivalent to $\mathcal{T}[\![\Delta]\!];\mathcal{T}[\![\Gamma]\!]\rho_\Delta \vdash_{\mathsf{T}} \mathcal{E}[\![\Lambda\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.e]\!]\rho_\Delta : \mathcal{T}[\![\forall\alpha_1 <: \tau_1, \ldots, \alpha_n <: \tau_n.\tau]\!]\rho_\Delta$. (We can replace $\mathcal{T}[\![\Gamma]\!]\rho'_\Delta$ by $\mathcal{T}[\![\Gamma]\!]\rho_\Delta$ by the assumption that Γ is a good context in Δ.) ∎

E Specialization in Practice

While Section 6 describes an interesting theoretical result (the ability to define a `toDnf` function whose type statically enforces a structural invariant without recourse to separate datatypes), it is not yet clear whether the methodology presented is usable in practice. In particular, pattern-matching on specialized types must be performed via application of the destructor functions, rather than via SML's built-in syntactic support for pattern-matching. The result is the "truly unreadable code" in Figure 6. While any use of pattern-matching can be desugared to applications of the destructor functions, we feel that a number of important aspects of the pattern-matching programming idom are seriously inhibited by the encoding in the previous section.

First, and perhaps most importantly, the destructor functions force the programmer to decompose a specialized type in a particular order; furthermore, the syntax provides no hint as to the correct order! Whereas a static-typing error will flag the use of the anonymous function `fn () => falseDnf` in the tuple slot for the `And` element of the specialization, nothing prevents us from interchanging `fn (f,g) => andDnfs (toDnf' f, toDnf' g)` and `fn (f,g) => orDnfs (toDnf' f, toDnf' g)` in the `toDnf` function of Figure 6. It is true that pattern-matching would not prevent us from performing the erroneous interchange, but the incongruity of the pattern `And (f,g)` followed by the expression `orDnfs (toDnf' f, toDnf' g)` in an arm of a `case` expression would certainly catch the eye in a way that the anonymous functions in a destructor function do not.

A simple solution to this immediate problem is to replace the tuple argument in the destructor functions with a record argument. The type for the `caseFmla` function would then have the following type:

```
val caseFmla : 'a AFmla ->
               {Var: string -> 'b,
                Not: CFmla -> 'b,
                True: unit -> 'b,
                And: CFmla * CFmla -> 'b,
                False: unit -> 'b,
                Or: CFmla * CFmla -> 'b} -> 'b
```

Using this function, the programmer is now free to discriminate and deconstruct elements of the specialization in any order. Furthermore, it is now manifest which element corresponds to each dispatch function.

Yet, this solution is still a far cry from the full power (in both expressivity and convenience) of pattern-matching. Examining Figure 6 reveals two particularly glaring assaults on readability that could be improved with pattern-matching, and cannot be improved by the previous solution. First, consider the `andDnfs` function. Informally, one can describe the intended behavior of the function in the following way: *consider both arguments in the `dnf` specialization: if either argument is a `False` element, return the other argument; if both arguments are `Or` elements, return the disjunction of the pairwise conjunction of the elements' arguments.* Unfortunately, the written code obscures this behavior. Reading the code, one might describe the actual behavior as follows: *consider the first argument in the `dnf` specialization: if it is a `False` element, return the second argument; if it is an `Or` element, then consider the second argument in the `dnf` specialization: if it is a `False` element, return the first argument; if it is an `Or` element, return the disjunction of the pairwise conjunction of the elements' arguments.* This highlights two missing aspects of pattern-matching: the ability to pattern-match simultaneously (via the nesting of datatype patterns within tuple patterns) and the ability to give a wild-card match.

Second, consider the `toDnf'` function. In this function, note the nested application of the `caseFmla` destructor function. Here, one misses the ability to write nested patterns, which would combine the two applications of destructor functions into one pattern match. Clearly, the solution presented above provides no help in these situations.

We seek, therefore, for a solution that brings the expressivity and convienence of pattern-matching to specialized types. As we stated earlier, we could accomplish the same static invariants as provided by specialized types by using distinct datatypes for each specialization and providing function to convert between them. Of course, this is rather inefficient. For example, converting a `dnf` specialization to a string via `(toString o dnfToFmla)` requires two complete traversals of the DNF formula and the allocation of an intermediate structure (of the same size as the original formula). As a compromise, we describe a middle-ground solution – one that uses distinct datatypes

```
signature FMLA_DT = sig
  include FMLA

  (* datatypes *)
  datatype ('a,'b,'c,'d,'e) DFmla = VarFmla of string | NotFmla of 'a
                                  | TrueFmla | AndFmla of 'b * 'c
                                  | FalseFmla | OrFmla of 'd * 'e
  datatype DAtom = VarAtom of string
  datatype 'a DLit = VarLit of string | NotLit of 'a
  datatype ('a, 'b) DConj = TrueConj | AndConj of 'a * 'b
  datatype ('a, 'b) DDnf = FalseDnf | OrDnf of 'a * 'b

  (* injections *)
  val injFmla : ('a AFmla,
                 'b AFmla, 'c AFmla,
                 'd AFmla, 'e AFmla) DFmla -> CFmla
  val injLit : 'a AAtom DLit -> CLit
  val injAtom : DAtom -> CAtom
  val injConj : ('a ALit, 'b AConj) DConj -> CConj
  val injDnf : ('a AConj, 'b ADnf) DDnf -> CDnf

  (* projections *)
  val prjFmla : 'a AFmla -> (CFmla,
                             CFmla, CFmla,
                             CFmla, CFmla) DFmla
  val prjLit : 'a ALit -> CAtom DLit
  val prjAtom : 'a AAtom -> DAtom
  val prjConj : 'a AConj -> (CLit, CConj) DConj
  val prjDnf : 'a ADnf -> (CConj, CDnf) DDnf

  (* Maps *)
  val mapFmla : (('a1 -> 'a2) *
                 ('b1 -> 'b2) * ('c1 -> 'c2) *
                 ('d1 -> 'd2) * ('e1 -> 'e2)) ->
                ('a1,'b1,'c1,'d1,'e1) DFmla ->
                ('a2,'b2,'c2,'d2,'e2) DFmla
  val mapAtom : DAtom -> DAtom
  val mapLit : ('a1 -> 'a2) -> 'a1 DLit -> 'a2 DLit
  val mapConj : (('a1 -> 'a2) * ('b1 -> 'b2)) -> ('a1,'b1) DConj -> ('a2,'b2) DConj
  val mapDnf : (('a1 -> 'a2) * ('b1 -> 'b2)) -> ('a1,'b1) DDnf -> ('a2,'b2) DDnf
end
```

Figure 7: The FMLA_DT signature

for their induced patterns and fine-grained coercions to localize coercions to and from the abstract specialized type and the datatypes.

We take as inspiration Wang and Murphy's recursion schemes [Wang and Murphy VII 2003]. Exploiting two-level types, which split an inductively defined type into a component that represents the structure of the type and a component that ties the recursive knot, recursion schemes provide a programming idiom that can hide the representation of an abstract type while still supporting pattern matching. Figures 7 and 8 give a signature and corresponding implementation of datatypes for the specializations of the formulas. (Again, while the quantity of code is large, it is also largely boilerplate code that can be mechanically generated from a declarative description of the specializations.) Note that the signature and implementation are written against the `FMLA` signature and `Fmla` structure; in particular, the `FMLA_DT` signature and `FmlaDT` structure do not require access to the unspecialized datatype. Therefore, we need not make any additional arguments about the safety of using the datatype interface to the specializations. That is, we cannot violate the invariants imposed

```
structure FmlaDT : FMLA_DT = struct
  open Fmla

  (* datatypes *)
  datatype ('a,'b,'c,'d,'e) DFmla = VarFmla of string | NotFmla of 'a
                                  | TrueFmla | AndFmla of 'b * 'c
                                  | FalseFmla | OrFmla of 'd * 'e
  datatype DAtom = VarAtom of string
  datatype 'a DLit = VarLit of string | NotLit of 'a
  datatype ('a, 'b) DConj = TrueConj | AndConj of 'a * 'b
  datatype ('a, 'b) DDnf = FalseDnf | OrDnf of 'a * 'b

  (* injections *)
  fun injFmla f = case f of VarFmla s => varFmla s
                          | NotFmla f => notFmla f
                          | TrueFmla => trueFmla
                          | AndFmla (f1, f2) => andFmla (f1, f2)
                          | FalseFmla => falseFmla
                          | OrFmla (f1, f2) => orFmla (f1, f2)
  fun injLit f = case f of VarLit s => varLit s
                         | NotLit f => notLit f
  fun injAtom f = case f of VarAtom s => varAtom s
  fun injConj f = case f of TrueConj => trueConj
                          | AndConj (f1, f2) => andConj (f1, f2)
  fun injDnf f = case f of FalseDnf => falseDnf
                         | OrDnf (f1, f2) => orDnf (f1, f2)

  (* projections *)
  fun prjFmla f = caseFmla f (VarFmla,NotFmla,
                              fn () => TrueFmla,AndFmla,
                              fn () => FalseFmla,OrFmla)
  fun prjLit f = caseLit f (VarLit,NotLit)
  fun prjAtom f = caseAtom f (VarAtom)
  fun prjConj f = caseConj f (fn () => TrueConj,AndConj)
  fun prjDnf f = caseDnf f (fn () => FalseDnf,OrDnf)

  (* Maps *)
  fun mapFmla (F1,F2,F3,F4,F5) f = case f of VarFmla s => VarFmla s
                                           | NotFmla f => NotFmla (F1 f)
                                           | TrueFmla => TrueFmla
                                           | AndFmla (f1, f2) => AndFmla (F2 f1, F3 f2)
                                           | FalseFmla => FalseFmla
                                           | OrFmla (f1, f2) => OrFmla (F4 f1, F5 f2)
  fun mapAtom f = case f of VarAtom s => VarAtom s
  fun mapLit F f = case f of VarLit s => VarLit s
                           | NotLit f => NotLit (F f)
  fun mapConj (F1,F2) f = case f of TrueConj => TrueConj
                                  | AndConj (f1, f2) => AndConj (F1 f1, F2 f2)
  fun mapDnf (F1,F2) f = case f of FalseDnf => FalseDnf
                                 | OrDnf (f1, f2) => OrDnf (F1 f1, F2 f2)
end
```

Figure 8: The `FmlaDT` structure

by the specializations by using the datatype interface.

As we did in Section 6, let us examine the different elements of the signature and their implementation.

E.1 Datatypes

The first part of the signature defines the datatypes used to represent specializations. The datatype for a specialization has a datatype constructor for each constructor in the specialization. Wherever the specialization references a specialization, we introduce a type variable, inducing a polymorphic datatype. The polymorphic type allows the datatype to represent unfoldings of the recursive specialization type with the abstract specialization type at the "leaves" of the structure. For instance, the `dnf` specialization

```
withspec dnf = False | Or of conj * dnf
```

becomes

```
datatype ('a, 'b) DDnf = FalseDnf
                       | OrDnf of 'a * 'b
```

replacing the references to the specializations `conj` and `dnf` with the polymorphic type variables `'a` and `'b`. Thus, the structure of a `dnf` specialization is given without reference to specific types for `conj` or `dnf`.

While this implementation is inspired by recursion schemes, it is at this point that we deviate significantly. Recursion schemes as described by Wang and Murphy [2003] are designed to support monomorphic recursive types, with a straightforward extension to polymorphic recursive types. However, it is less clear how to extend the idiom to mutually recursive types, which sometimes arise in specializations (e.g., the `boolexp` and `intexp` specializations given in Section 6). Hence, we have chosen to allow each reference to a specialization to be typed independently. Clearly we do not wish to identify `conj` and `dnf` in the `dnf` datatype above, because the two specializations are distinct. It is debatable whether all occurences of the `fmla` specialization within the `fmla` specialization should be identified; that is, whether we should introduce

```
datatype ('a,'b,'c,'d,'e) DFmla =
    VarFmla of string | NotFmla of 'a
  | TrueFmla | AndFmla of 'b * 'c
  | FalseFmla | OrFmla of 'd * 'e
```

or

```
datatype 'a DFmla =
    VarFmla of string | NotFmla of 'a
  | TrueFmla | AndFmla of 'a * 'a
  | FalseFmla | OrFmla of 'a * 'a
```

The latter is closer to recursion schemes (and hence could be given a simple categorical interface as described by Wang and Murphy [2003]). For pragmatic reasons, we have instead adopted the former, because it gives finer grained control over coercions to and from the datatype. However, as the entire implementation does not require access to the unspecialized datatype, either or both could be given without difficulty.

The polymorphic type variables allows the datatype to represent arbitrary, finite unrollings of the specialized type. For example, the type (CConj, CDnf DDnf) DDnf corresponds to unrolling the dnf specialization into the DDnf datatype once at the top-level and once again at the second argument to the And constructor. Hence, it has as elements FalseDnf, OrDnf (e1, FalseDnf), and OrDnf (e1, OrDnf (e2, e3)) for any e1 and e2 of type CConj and e3 of type CDnf.

E.2 Injections

For every specialization, the interface provides a function for injecting from a specialization's datatype into the specialization itself. In particular, the injection is from one top-level unrolling of the specialization into the datatype back into the specialization. The implementation is straightforward: map each datatype constructor to the corresponding specialization constructor. For the DDnf datatype, this yields

```
fun injDnf f =
  case f of FalseDnf => falseDnf
          | OrDnf (f1, f2) => orDnf (f1, f2)
```

with the type ('a AConj, 'b ADnf) DDnf → CDnf. We find injections to be used infrequently in practice. One usually wishes to build values at the specialization type, and the specialization constructors are better suited for this purpose than injecting from the specialization's datatype. Hence, we will not use injections in our examples.

E.3 Projections

Much more practical are the projection functions, which unroll a specialization into the datatype representation. Again, the unrolling is from the specialization to a single top-level unrolling of the specialization. The implementation builds upon the destructor functions, by mapping each dispatch function to the corresponding datatype constructor. For the DDnf datatype, this yields

```
fun prjDnf f =
  caseDnf f (fn () => FalseDnf,
             fn (f1,f2) => OrDnf (f1,f2))
```

with the type 'a ADnf → (CConj, CDnf) DDnf.

E.4 Maps

One final useful family of functions are the structure preserving maps. These functions, similar in flavor to the familiar map on polymorphic lists, apply a function to each polymorphic element of a structure, but otherwise leave the structure's "shape" intact. For instance, the map

```
fun mapLit F f =
  case f of VarLit s => VarLit s
          | NotLit f => NotLit (F f)
```

```
fun andConjs (f: CConj, g: CConj): CConj =
  case prjConj f of TrueConj => g
                  | AndConj (f1,f2) => andConj (f1, andConjs (f2, g))
fun orDnfs (f: CDnf, g: CDnf): CDnf =
  case prjDnf f of FalseDnf => g
                 | OrDnf (f1,f2) => orDnf (f1, orDnfs (f2, g))
fun andConjDnf (f: CConj, g: CDnf): CDnf =
  case prjDnf g of FalseDnf => falseDnf
                 | OrDnf (g1, g2) => orDnf (andConjs (f, g1),
                                            andConjDnf (f, g2))
fun andDnfs (f: CDnf, g: CDnf): CDnf =
  case (prjDnf f, prjDnf g) of
    (FalseDnf, _) => falseDnf
  | (_, FalseDnf) => falseDnf
  | (OrDnf (f1,f2), OrDnf (g1,g2)) => orDnf (andConjs (f1, g1),
                                      orDnfs (andConjDnf (f1, g2),
                                      orDnfs (andConjDnf (g1, f2),
                                              andDnfs (f2, g2))))
fun litToDnf (f: 'a ALit): CDnf = orDnf (andConj (f, trueConj), falseDnf)
fun toDnf (f: 'a AFmla): CDnf = let
  fun toDnf' (f: CFmla): CDnf =
    case mapFmla (prjFmla, id, id, id, id) (prjFmla f) of
      VarFmla s => litToDnf (varAtom s)
    | NotFmla (VarFmla s) => litToDnf (notLit (varAtom s))
    | NotFmla (NotFmla f) => toDnf' f
    | NotFmla TrueFmla => toDnf' falseFmla
    | NotFmla (AndFmla (f,g)) => toDnf' (orFmla (notFmla f, notFmla g))
    | NotFmla FalseFmla => toDnf' trueFmla
    | NotFmla (OrFmla (f,g)) => toDnf' (andFmla (notFmla f, notFmla g))
    | TrueFmla => orDnf (trueConj, falseDnf)
    | AndFmla (f,g) => andDnfs (toDnf' f, toDnf' g)
    | FalseFmla => falseDnf
    | OrFmla (f,g) => orDnfs (toDnf' f, toDnf' g)
in toDnf' (coerceFmla f) end
```

Figure 9: The `toDnf` function (via datatype interface)

transforms an `'a1 DLit` to an `'a2 DLit` via a function `F` of type `'a1` → `'a2`. Since the polymorphic elements of a specialization datatype correspond to the nested specializations used by that datatype, maps are useful for localizing unfoldings of a specialization for nested pattern matching.

E.5 Example

Figure 9 reproduces the code from Figure 6 using the datatype interface to the formula specializations. While some may argue that the syntactic differences are minor, those differences are important ones. We have addressed all the shortcomings described at the beginning of this section. All of the functions are written in a familar, readable pattern-matching style. In particular, the `andDnfs` function uses simultaneous pattern-matching and wild-card matches to efficiently implement the textual description described previously. Also, the `toDnf'` function uses nested-patterns to fold all the branches into a single `case` expression. Note the use of the `mapFmla` function to unfold only the `CFmla` under the `NotFmla` constructor, while leaving all other `fmla` specializations folded. The expression discriminated by the `case` has the following type:

```
((CFmla, CFmla, CFmla, CFmla, CFmla) DFmla,
 CFmla, CFmla, CFmla, CFmla) DFmla
```

Finally, a word about the efficiency of the compiled code. In the presence of cross module inlining, smart representation decisions, and some local constant folding, the overhead of projections from a specialization type to a datatype can be almost entirely eliminated. Specifically, when projections appear directly as the expression discriminated by a `case`, then a compiler can easily fold the `case` expression "buried" in the projection function into the outer `case` expression.

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