Anhang A

Typentheorie: Syntax, Semantik, Inferenzregeln

A.1 Parametertypen

`variable` : Variablennamen (ML Datentyp `var`)
Zulässige Elemente sind Zeichenketten der Form [a–z A–Z 0–9 _ - %]$^+$

`natural` : Natürliche Zahlen einschließlich der Null (ML Datentyp `int`).
Zulässige Elemente sind Zeichenketten der Form $0 + [1-9][0-9]^*$.

`token` : Zeichenketten für Namen (ML Datentyp `tok`).
Zulässige Elemente bestehen aus allen Symbolen des NuPRL Zeichensatzes mit Ausnahme der Kontrollsymbole.

`string` : Zeichenketten für Texte (ML Datentyp `string`).
Zulässige Elemente bestehen aus allen Symbolen des NuPRL Zeichensatzes mit Ausnahme der Kontrollsymbole.

`level-expression` : Ausdrücke für das Level eines Typuniversums (ML Datentyp `level_exp`)
Die genaue Syntax wird in Abschnitt **??** bei der Diskussion der Universenhierarchie besprochen.

Die Namen für Parametertypen dürfen durch ihre ersten Buchstaben abgekürzt werden.

A.2 Operatorentabelle

kanonisch (Typen)	(Elemente)	nichtkanonisch
	var$\{x:\mathtt{v}\}()$ x	
int$\{\}()$ $\mathbb{Z}$	**natnum**$\{n:\mathtt{n}\}()$ **minus**$\{\}($**natnum**$\{n:\mathtt{n}\}())$ n $-n$	**ind**$\{\}(\boxed{u};\, x,f_x.s;\, base;\, y,f_y.t)$ **minus**$\{\}(\boxed{u})$, **add**$\{\}(\boxed{u};\boxed{v})$, **sub**$\{\}(\boxed{u};\boxed{v})$ **mul**$\{\}(\boxed{u};\boxed{v})$, **div**$\{\}(\boxed{u};\boxed{v})$, **rem**$\{\}(\boxed{u};\boxed{v})$ **int_eq**$(\boxed{u};\, \boxed{v};\, s;\, t)$, **less**$(\boxed{u};\, \boxed{v};\, s;\, t)$ ind$(\boxed{u};\, x,f_x.s;\, base;\, y,f_y.t)$ $-\boxed{u}$, $\boxed{u}+\boxed{v}$, $\boxed{u}-\boxed{v}$ $\boxed{u}*\boxed{v}$, $\boxed{u}\div\boxed{v}$, $\boxed{u}$ rem $\boxed{v}$ if $\boxed{u}=\boxed{v}$ then s else t, if $\boxed{u}<\boxed{v}$ then s else t
lt$\{\}(u;v)$ $u<v$	**Axiom**$\{\}()$ Axiom	
void$\{\}()$ void		**any**$\{\}(e)$ any(e)
Atom$\{\}()$ Atom	**token**$\{string:\mathtt{t}\}()$ ¨*string*¨	**atom_eq**$(\boxed{u};\, \boxed{v};\, s;\, t)$ if $\boxed{u}=\boxed{v}$ then s else t
U$\{j:\mathtt{l}\}()$ U_j	(alle kanonischen Typen)	
fun$\{\}(S;\, x.T)$ $x:S\rightarrow T$	**lam**$\{\}(x.t)$ $\lambda x.t$	**apply**$\{\}(\boxed{f};t)$ $\boxed{f}\ t$
prod$\{\}(S;\, x.T)$ $x:S\times T$	**pair**$\{\}(s;t)$ $\langle s,t\rangle$	**spread**$\{\}(\boxed{e};\, x,y.u)$ let $\langle x,y\rangle=\boxed{e}$ in u
union$\{\}(S;T)$ $S+T$	**inl**$\{\}(s)$, **inr**$\{\}(t)$ inl(s), inr(t)	**decide**$\{\}(\boxed{e};\, x.u;\, y.v)$ case $\boxed{e}$ of inl$(x)\mapsto u$ | inr$(y)\mapsto v$
equal$\{\}(s;t;T)$ $s=t\in T$	**Axiom**$\{\}()$ Axiom	
list$\{\}(T)$ T list	**nil**$\{\}()$, **cons**$\{\}(t;l)$ [], $t.l$	**list_ind**$\{\}(\boxed{s};\, base;\, x,l,f_{xl}.t)$ list_ind$(\boxed{s};\, base;\, x,l,f_{xl}.t)$
set$\{\}(S;\, x.T)$ $\{x:S \mid T\}$	(Elemente von S)	
quotient$\{\}(T;\, x,y.E)$ $x,y : T//E$	(Elemente von T)	
rec$\{\}(X.T_X)$ rectype $X=T_X$	(Elemente gemäß T_X)	**rec_ind**$\{\}(\boxed{e};\, f,x.t)$ let* $f(x)=t$ in $f(\boxed{e})$
pfun$\{\}(S;\, T)$ $S\not\rightarrow T$	**fix**$\{\}(f,x.t)$ letrec $f(x)=t$	**dom**$\{\}(\boxed{f})$, **apply_p**$\{\}(\boxed{f};t)$ dom$(\boxed{f})$, $\boxed{f}\ (t)$

Man beachte, daß in der Implementierung von NuPRL 4.0 die konkreten Namen und Displayformen zum Teil etwas anders sind als hier angegeben. An der Anpassung wird gearbeitet.

A.3 Redizes und Kontrakta

Redex		Kontraktum
$(\lambda x . u)\ t$	$\xrightarrow{\beta}$	$u[t/x]$
let $\langle x, y\rangle = \langle s, t\rangle$ in u	$\xrightarrow{\beta}$	$u[s,t\,/\,x,y]$
case inl(s) of inl$(x) \mapsto u$ | inr$(y) \mapsto v$	$\xrightarrow{\beta}$	$u[s/x]$
case inr(t) of inl$(x) \mapsto u$ | inr$(y) \mapsto v$	$\xrightarrow{\beta}$	$v[t/y]$
ind$(0;\ x, f_x . s;\ base;\ y, f_y . t)$	$\xrightarrow{\beta}$	$base$
ind$(n;\ x, f_x . s;\ base;\ y, f_y . t)$	$\xrightarrow{\beta}$	$t[n,$ind$(n{-}1;\ x, f_x . s;\ base;\ y, f_y . t)\ /x, f_x]$, $(n{>}0)$
ind$(-n;\ x, f_x . s;\ base;\ y, f_y . t)$	$\xrightarrow{\beta}$	$s[-n,$ind$(-n{+}1;\ x, f_x . s;\ base;\ y, f_y . t)\ /y, f_y]$, $(n{>}0)$
$-i$	$\xrightarrow{\beta}$	*Die Negation von i (als Zahl)*
i+j	$\xrightarrow{\beta}$	*Die Summe von i und j*
i-j	$\xrightarrow{\beta}$	*Die Differenz von i und j*
i*j	$\xrightarrow{\beta}$	*Das Produkt von i und j*
$i \div j$	$\xrightarrow{\beta}$	0, *falls j=0; ansonsten die Integer-Division von i und j*
i rem j	$\xrightarrow{\beta}$	0, *falls j=0; ansonsten der Rest der Division von i und j*
if i=j then s else t	$\xrightarrow{\beta}$	s, *falls $i = j$; ansonsten t*
if i<j then s else t	$\xrightarrow{\beta}$	s, *falls $i < j$; ansonsten t*
list_ind([] ; $base;\ x, l, f_{xl} . t$)	$\xrightarrow{\beta}$	$base$
list_ind($s . u;\ base;\ x, l, f_{xl} . t$)	$\xrightarrow{\beta}$	$t[s, u,$list_ind$(u;\ base;\ x, l, f_{xl} . t)\ /x, l, f_{xl}]$
if u=v then s else t	$\xrightarrow{\beta}$	s, *falls $u = v$; ansonsten t*
let* $f(x) = t$ in $f(e)$	$\xrightarrow{\beta}$	$t[\lambda y .$ let* $f(x) = t$ in $f(y),\ e\ /\ f, x]$
(letrec $f(x) = t$) (u)	$\xrightarrow{\beta}$	$t[$letrec $f(x) = t,\ u\ /\ f, x]$
dom(letrec $f(x) = t$)	$\xrightarrow{\beta}$	$\lambda x .$ rectype $F = \mathcal{E}[[t]]$

A.4 Urteile

A.4.1 Typsemantik

$\mathbb{Z} = \mathbb{Z}$		
$i_1 {<} j_1 = i_2 {<} j_2$	falls	$i_1 = i_2 \in \mathbb{Z}$ und $j_1 = j_2 \in \mathbb{Z}$
$\mathsf{void} = \mathsf{void}$		
$\mathsf{Atom} = \mathsf{Atom}$		
$\mathsf{U}_{j_1} = \mathsf{U}_{j_2}$	falls	$j_1 = j_2$ *(als natürliche Zahl)*
$x_1{:}S_1 \to T_1 = x_2{:}S_2 \to T_2$	falls	$S_1 = S_2$ und $T_1[s_1/x_1] = T_2[s_2/x_2]$ für alle Terme s_1, s_2 mit $s_1 = s_2 \in S_1$.
$T = S_2 \to T_2$	falls	$T = x_2{:}S_2 \to T_2$ für ein beliebiges $x_2 \in \mathcal{V}$.
$S_1 \to T_1 = T$	falls	$x_1{:}S_1 \to T_1 = T$ für ein beliebiges $x_1 \in \mathcal{V}$.
$x_1{:}S_1 \times T_1 = x_2{:}S_2 \times T_2$	falls	$S_1 = S_2$ und $T_1[s_1/x_1] = T_2[s_2/x_2]$ für alle Terme s_1, s_2 mit $s_1 = s_2 \in S_1$.
$T = S_2 \times T_2$	falls	$T = x_2{:}S_2 \times T_2$ für ein beliebiges $x_2 \in \mathcal{V}$.
$S_1 \times T_1 = T$	falls	$x_1{:}S_1 \times T_1 = T$ für ein beliebiges $x_1 \in \mathcal{V}$.
$S_1 + T_1 = S_2 + T_2$	falls	$S_1 = S_2$ und $T_1 = T_2$.
$s_1 \mathsf{=} t_1 \in T_1 = s_2 \mathsf{=} t_2 \in T_2$	falls	$T_1 = T_2$ und $s_1 = s_2 \in T_1$ und $t_1 = t_2 \in T_1$.
$T_1\ \mathsf{list} = T_2\ \mathsf{list}$	falls	$T_1 = T_2$
$\{x_1 : S_1 \mid T_1\} = \{x_2 : S_2 \mid T_2\}$	falls	$S_1 = S_2$ und es gibt eine Variable x, die weder in T_1 noch in T_2 vorkommt, und zwei Terme p_1 und p_2 mit der Eigenschaft $p_1 \in \forall x{:}S_1 .\ T_1[x/x_1] \Rightarrow T_2[x/x_2]$ und $p_2 \in \forall x{:}S_1 .\ T_2[x/x_2] \Rightarrow T_1[x/x_1]$.
$T = \{S_2 \mid T_2\}$	falls	$T = \{x_2 : S_2 \mid T_2\}$ für ein beliebiges $x_2 \in \mathcal{V}$.
$\{S_1 \mid T_1\} = T$	falls	$\{x_1 : S_1 \mid T_1\} = T$ für ein beliebiges $x_1 \in \mathcal{V}$.
$x_1, y_1 : T_1 /\!/ E_1 = x_2, y_2 : T_2 /\!/ E_2$	falls	$T_1 = T_2$ und es gibt (verschiedene) Variablen x, y, z, die weder in E_1 noch in E_2 vorkommen, und Terme p_1, p_2, r,s und t mit der Eigenschaft $p_1 \in \forall x{:}T_1 . \forall y{:}T_1 .\ E_1[x,y/x_1,y_1] \Rightarrow E_2[x,y/x_2,y_2]$ und $p_2 \in \forall x{:}T_1 . \forall y{:}T_1 .\ E_2[x,y/x_2,y_2] \Rightarrow E_1[x,y/x_1,y_1]$ und $r \in \forall x{:}T_1 . E_1[x,x/x_1,y_1]$ und $s \in \forall x{:}T_1 . \forall y{:}T_1 .\ E_1[x,y/x_1,y_1] \Rightarrow E_1[y,x/x_1,y_1]$ und $t \in \forall x{:}T_1 . \forall y{:}T_1 . \forall z{:}T_1 .$ $E_1[x,y/x_1,y_1] \Rightarrow E_1[y,z/x_1,y_1] \Rightarrow E_1[x,z/x_1,y_1]$
$\mathsf{rectype}\ X_1 = T_{X1} = \mathsf{rectype}\ X_2 = T_{X2}$	falls	$T_{X1}[X/X_1] = T_{X2}[X/X_2]$ für alle Typen X
$S_1 \not\to T_1 = S_2 \not\to T_2$	falls	$S_1 = S_2$ und $T_1 = T_2$

A.4.2 Elementsemantik – Universen

Elementsemantik – Universen		
$\mathbb{Z} = \mathbb{Z} \in \mathsf{U}_j$		
$i_1 < j_1 = i_2 < j_2 \in \mathsf{U}_j$	falls	$i_1 = i_2 \in \mathbb{Z}$ und $j_1 = j_2 \in \mathbb{Z}$
$\mathsf{void} = \mathsf{void} \in \mathsf{U}_j$		
$\mathsf{Atom} = \mathsf{Atom} \in \mathsf{U}_j$		
$\mathsf{U}_{j_1} = \mathsf{U}_{j_2} \in \mathsf{U}_j$	falls	$j_1 = j_2 < j$ *(als natürliche Zahl)*
$x_1{:}S_1 \to T_1 = x_2{:}S_2 \to T_2 \in \mathsf{U}_j$	falls	$S_1 = S_2 \in \mathsf{U}_j$ und $T_1[s_1/x_1] = T_2[s_2/x_2] \in \mathsf{U}_j$ für alle Terme s_1, s_2 mit $s_1 = s_2 \in S_1$.
$T = S_2 \to T_2 \in \mathsf{U}_j$	falls	$T = x_2{:}S_2 \to T_2 \in \mathsf{U}_j$ für ein beliebiges $x_2 \in \mathcal{V}$.
$S_1 \to T_1 = T \in \mathsf{U}_j$	falls	$x_1{:}S_1 \to T_1 = T \in \mathsf{U}_j$ für ein beliebiges $x_1 \in \mathcal{V}$.
$x_1{:}S_1 \times T_1 = x_2{:}S_2 \times T_2 \in \mathsf{U}_j$	falls	$S_1 = S_2 \in \mathsf{U}_j$ und $T_1[s_1/x_1] = T_2[s_2/x_2] \in \mathsf{U}_j$ für alle Terme s_1, s_2 mit $s_1 = s_2 \in S_1$.
$T = S_2 \times T_2 \in \mathsf{U}_j$	falls	$T = x_2{:}S_2 \times T_2 \in \mathsf{U}_j$ für ein beliebiges $x_2 \in \mathcal{V}$.
$S_1 \times T_1 = T \in \mathsf{U}_j$	falls	$x_1{:}S_1 \times T_1 = T \in \mathsf{U}_j$ für ein beliebiges $x_1 \in \mathcal{V}$.
$S_1 + T_1 = S_2 + T_2 \in \mathsf{U}_j$	falls	$S_1 = S_2 \in \mathsf{U}_j$ und $T_1 = T_2 \in \mathsf{U}_j$.
$s_1 = t_1 \in T_1 \;=\; s_2 = t_2 \in T_2 \in \mathsf{U}_j$	falls	$T_1 = T_2 \in \mathsf{U}_j$ und $s_1 = s_2 \in T_1$ und $t_1 = t_2 \in T_1$.
$T_1\ \mathsf{list} = T_2\ \mathsf{list} \in \mathsf{U}_j$	falls	$T_1 = T_2 \in \mathsf{U}_j$
$\{x_1 : S_1 \mid T_1\} = \{x_2 : S_2 \mid T_2\} \in \mathsf{U}_j$	falls	$S_1 = S_2 \in \mathsf{U}_j$ und $T_1[s/x_1] \in \mathsf{U}_j$ sowie $T_2[s/x_2] \in \mathsf{U}_j$ gilt für alle Terme s mit $s \in S_1$ und es gibt eine Variable x, die weder in T_1 noch in T_2 vorkommt, und zwei Terme p_1 und p_2 mit der Eigenschaft $p_1 \in \forall x{:}S_1 .\, T_1[x/x_1] \Rightarrow T_2[x/x_2]$ und $p_2 \in \forall x{:}S_1 .\, T_2[x/x_2] \Rightarrow T_1[x/x_1]$
$T = \{S_2 \mid T_2\} \in \mathsf{U}_j$	falls	$T = \{x_2 : S_2 \mid T_2\} \in \mathsf{U}_j$ für ein beliebiges $x_2 \in \mathcal{V}$.
$\{S_1 \mid T_1\} = T \in \mathsf{U}_j$	falls	$\{x_1 : S_1 \mid T_1\} = T \in \mathsf{U}_j$ für ein beliebiges $x_1 \in \mathcal{V}$.
$x_1, y_1 : T_1 /\!/ E_1 = x_2, y_2 : T_2 /\!/ E_2 \in \mathsf{U}_j$	falls	$T_1 = T_2 \in \mathsf{U}_j$ und für alle Terme s, t mit $s \in T_1$ und $t \in T_1$ gilt $E_1[s,t/x_1,y_1] \in \mathsf{U}_j$ sowie $E_2[s,t/x_2,y_2] \in \mathsf{U}_j$ und es gibt (verschiedene) Variablen x, y, z, die weder in E_1 noch in E_2 vorkommen, und Terme p_1, p_2, r,s und t mit der Eigenschaft $p_1 \in \forall x{:}T_1 .\, \forall y{:}T_1 .\, E_1[x,y/x_1,y_1] \Rightarrow E_2[x,y/x_2,y_2]$ und $p_2 \in \forall x{:}T_1 .\, \forall y{:}T_1 .\, E_2[x,y/x_2,y_2] \Rightarrow E_1[x,y/x_1,y_1]$ und $r \in \forall x{:}T_1 .\, E_1[x,x/x_1,y_1]$ und $s \in \forall x{:}T_1 .\, \forall y{:}T_1 .\, E_1[x,y/x_1,y_1] \Rightarrow E_1[y,x/x_1,y_1]$ und $t \in \forall x{:}T_1 .\, \forall y{:}T_1 .\, \forall z{:}T_1 .$ $E_1[x,y/x_1,y_1] \Rightarrow E_1[y,z/x_1,y_1] \Rightarrow E_1[x,z/x_1,y_1]$
$\mathsf{rectype}\ X_1 = T_{X1} = \mathsf{rectype}\ X_2 = T_{X2} \in \mathsf{U}_j$	falls	$T_{X1}[X/X_1] = T_{X2}[X/X_2] \in \mathsf{U}_j$ für alle Terme X mit $X \in \mathsf{U}_j$
$S_1 \not\to T_1 = S_2 \not\to T_2 \in \mathsf{U}_j$	falls	$S_1 = S_2 \in \mathsf{U}_j$ und $T_1 = T_2 \in \mathsf{U}_j$

A.4.3 Elementsemantik

$i = i \in \mathbb{Z}$		
$\mathsf{Axiom} = \mathsf{Axiom} \in s$<$t$	falls	es ganze Zahlen i und j gibt, wobei i kleiner als j ist, für die gilt $s \xrightarrow{l} i$ und $t \xrightarrow{l} j$
$s = t \in \mathsf{void}$		*gilt niemals !*
$\ddot{}string\ddot{} = \ddot{}string\ddot{} \in \mathsf{Atom}$		
$\lambda x_1.t_1 = \lambda x_2.t_2 \in x{:}S{\to}T$	falls	$x{:}S{\to}T$ Typ und $t_1[s_1/x_1] = t_2[s_2/x_2] \in T[s_1/x]$ für alle Terme s_1, s_2 mit $s_1{=}s_2 \in S$.
$\langle s_1, t_1\rangle = \langle s_2, t_2\rangle \in x{:}S{\times}T$	falls	$x{:}S{\times}T$ Typ und $s_1{=}s_2 \in S$ und $t_1{=}t_2 \in T[s_1/x]$.
$\mathsf{inl}(s_1) = \mathsf{inl}(s_2) \in S{+}T$	falls	$S{+}T$ Typ und $s_1{=}s_2 \in S$.
$\mathsf{inr}(t_1) = \mathsf{inr}(t_2) \in S{+}T$	falls	$S{+}T$ Typ und $t_1{=}t_2 \in T$.
$\mathsf{Axiom} = \mathsf{Axiom} \in s = t \in T$	falls	$s{=}t \in T$
$[] = [] \in T\ \mathsf{list}$	falls	T Typ
$t_1.l_1 = t_2.l_2 \in T\ \mathsf{list}$	falls	T Typ und $t_1 = t_2 \in T$ und $l_1 = l_2 \in T\ \mathsf{list}$
$s = t \in \{x{:}S \mid T\}$	falls	$\{x{:}S \mid T\}$ Typ und $s = t \in S$ und es gibt einen Term p mit der Eigenschaft $p \in T[s/x]$
$s = t \in x, y : T//E$	falls	$x, y : T//E$ Typ und $s \in T$ und $t \in T$ und es gibt einen Term p mit der Eigenschaft $p \in E[s, t/x, y]$
$s = t \in \mathsf{rectype}\ X = T_X$	falls	$\mathsf{rectype}\ X = T_X$ Typ und $s = t \in T_X[\mathsf{rectype}\ X = T_X\ /\ X]$
$\mathsf{letrec}\ f_1(x_1) = t_1 = \mathsf{letrec}\ f_2(x_2) = t_2 \in S{\not\to}T$	falls	$S{\not\to}T$ Typ und für alle Terme s_1 und s_2 mit $s_1{=}s_2 \in S$ $\{x{:}S \mid \mathsf{dom}(\mathsf{letrec}\ f_1(x_1) = t_1)(x)\}$ $= \{x{:}S \mid \mathsf{dom}(\mathsf{letrec}\ f_2(x_2) = t_2)(x)\}$ und $t_1[\mathsf{letrec}\ f_1(x_1) = t_1,\ s_1\ /\ f_1, x_1]$ $= t_2[\mathsf{letrec}\ f_2(x_2) = t_2,\ s_2\ /\ f_2, x_2] \in T$

A.5 Inferenzregeln

A.5.1 Ganze Zahlen

$\Gamma \vdash \mathbb{Z} \in \mathsf{U}_j$ ⌊Ax⌋
by intEq

$\Gamma \vdash n \in \mathbb{Z}$ ⌊Ax⌋
by natnumEq

$\Gamma \vdash \mathsf{ind}(u_1; x_1, f_{x1}.s_1; base_1; y_1, f_{y1}.t_1)$
$= \mathsf{ind}(u_2; x_2, f_{x2}.s_2; base_2; y_2, f_{y2}.t_2)$
$\in T[u_1/z]$ ⌊Ax⌋
by indEq z T
$\Gamma \vdash u_1\texttt{=}u_2$ ⌊Ax⌋
$\Gamma,\ x{:}\mathbb{Z},\ v{:}\,x\texttt{<}0,\ f_x{:}T[(x\texttt{+}1)/z]$
$\vdash s_1[x, f_x / x_1, f_{x1}] = s_2[x, f_x / x_2, f_{x2}] \in T[x/z]$
⌊Ax⌋
$\Gamma \vdash base_1 = base_2 \in T[0/z]$ ⌊Ax⌋
$\Gamma,\ x{:}\mathbb{Z},\ v{:}\,0\texttt{<}x,\ f_x{:}T[(x\texttt{-}1)/z]$
$\vdash t_1[x, f_x / y_1, f_{y1}] = t_2[x, f_x / y_2, f_{y2}] \in T[x/z]$ ⌊Ax⌋

$\Gamma \vdash \texttt{-}s_1 = \texttt{-}s_2$ ⌊Ax⌋
by minusEq
$\Gamma \vdash s_1 = s_2$ ⌊Ax⌋

$\Gamma \vdash s_1\texttt{+}t_1 = s_2\texttt{+}t_2$ ⌊Ax⌋
by addEq
$\Gamma \vdash s_1 = s_2$ ⌊Ax⌋
$\Gamma \vdash t_1 = t_2$ ⌊Ax⌋

$\Gamma \vdash s_1\texttt{-}t_1 = s_2\texttt{-}t_2$ ⌊Ax⌋
by subEq
$\Gamma \vdash s_1 = s_2$ ⌊Ax⌋
$\Gamma \vdash t_1 = t_2$ ⌊Ax⌋

$\Gamma \vdash s_1\texttt{*}t_1 = s_2\texttt{*}t_2$ ⌊Ax⌋
by mulEq
$\Gamma \vdash s_1 = s_2$ ⌊Ax⌋
$\Gamma \vdash t_1 = t_2$ ⌊Ax⌋

$\Gamma \vdash s_1 \div t_1 = s_2 \div t_2$ ⌊Ax⌋
by divEq
$\Gamma \vdash s_1 = s_2$ ⌊Ax⌋
$\Gamma \vdash t_1 = t_2$ ⌊Ax⌋
$\Gamma \vdash t_1 \neq 0$ ⌊Ax⌋

$\Gamma \vdash s_1 \,\mathsf{rem}\, t_1 = s_2 \,\mathsf{rem}\, t_2$ ⌊Ax⌋
by remEq
$\Gamma \vdash s_1 = s_2$ ⌊Ax⌋
$\Gamma \vdash t_1 = t_2$ ⌊Ax⌋
$\Gamma \vdash t_1 \neq 0$ ⌊Ax⌋

$\Gamma \vdash \mathsf{if}\ u_1\texttt{=}v_1\ \mathsf{then}\ s_1\ \mathsf{else}\ t_1$
$= \mathsf{if}\ u_2\texttt{=}v_2\ \mathsf{then}\ s_2\ \mathsf{else}\ t_2 \in T$ ⌊Ax⌋
by int_eqEq
$\Gamma \vdash u_1\texttt{=}u_2$ ⌊Ax⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** n⌋
by natnumI n

$\Gamma,\ z{:}\mathbb{Z},\ \Delta \vdash C$
⌊**ext** $\mathsf{ind}(z; x, f_x.s[\mathsf{Axiom}/v]; base; x, f_x.t[\mathsf{Axiom}/v])$⌋
by intE i
$\Gamma,\ z{:}\mathbb{Z},\ \Delta,\ x{:}\mathbb{Z},\ v{:}\,x\texttt{<}0,\ f_x{:}C[(x\texttt{+}1)/z]$
$\vdash C[x/z]$ ⌊**ext** s⌋
$\Gamma,\ z{:}\mathbb{Z},\ \Delta \vdash C[0/z]$ ⌊**ext** $base$⌋
$\Gamma,\ z{:}\mathbb{Z},\ \Delta,\ x{:}\mathbb{Z},\ v{:}\,0\texttt{<}x,\ f_x{:}C[(x\texttt{-}1)/z]$
$\vdash C[x/z]$ ⌊**ext** t⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** $\texttt{-}s$⌋
by minusI
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** s⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** $s\texttt{+}t$⌋
by addI
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** s⌋
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** t⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** $s\texttt{-}t$⌋
by subI
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** s⌋
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** t⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** $s\texttt{*}t$⌋
by mulI
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** s⌋
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** t⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** $s \div t$⌋
by divI
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** s⌋
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** t⌋

$\Gamma \vdash \mathbb{Z}$ ⌊**ext** $s \,\mathsf{rem}\, t$⌋
by remI
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** s⌋
$\Gamma \vdash \mathbb{Z}$ ⌊**ext** t⌋

$\Gamma \vdash \mathsf{if}\ u_1\texttt{<}v_1\ \mathsf{then}\ s\ \mathsf{else}\ t$
$= \mathsf{if}\ u_2\texttt{<}v_2\ \mathsf{then}\ s_2\ \mathsf{else}\ t_2 \in T$ ⌊Ax⌋
by lessEq
$\Gamma \vdash u_1\texttt{=}u_2$ ⌊Ax⌋

$\Gamma \vdash \mathsf{ind}(i;\, x, f_x.s;\, base;\, y, f_y.t) = t_2 \in T$ [Ax]
by <u>indRedDown</u>
$\Gamma \vdash t[i, \mathsf{ind}(i{+}1;\, x, f_x.s;\, base;\, y, f_y.t)\, /x, f_x]$
$= t_2 \in T$ [Ax]
$\Gamma \vdash i<0$ [Ax]

$\Gamma \vdash \mathsf{ind}(i;\, x, f_x.s;\, base;\, y, f_y.t) = t_2 \in T$ [Ax]
by <u>indRedUp</u>
$\Gamma \vdash t[i, \mathsf{ind}(i{-}1;\, x, f_x.s;\, base;\, y, f_y.t)\, /y, f_y]$
$= t_2 \in T$ [Ax]
$\Gamma \vdash 0<i$ [Ax]

$\Gamma \vdash \mathsf{ind}(i;\, x, f_x.s;\, base;\, y, f_y.t) = t_2 \in T$ [Ax]
by <u>indRedBase</u>
$\Gamma \vdash base = t_2 \in T$ [Ax]
$\Gamma \vdash i{=}0$ [Ax]

$\Gamma \vdash \mathsf{if}\ u{=}v\ \mathsf{then}\ s\ \mathsf{else}\ t = t_2 \in T$ [Ax]
by <u>int_eqRedT</u>
$\Gamma \vdash s = t_2 \in T$ [Ax]
$\Gamma \vdash u{=}v$ [Ax]

$\Gamma \vdash \mathsf{if}\ u{=}v\ \mathsf{then}\ s\ \mathsf{else}\ t = t_2 \in T$ [Ax]
by <u>int_eqRedF</u>
$\Gamma \vdash t = t_2 \in T$ [Ax]
$\Gamma \vdash u \neq v$ [Ax]

$\Gamma \vdash \mathsf{if}\ u{<}v\ \mathsf{then}\ s\ \mathsf{else}\ t = t_2 \in T$ [Ax]
by <u>lessRedT</u>
$\Gamma \vdash s = t_2 \in T$ [Ax]
$\Gamma \vdash u{<}v$ [Ax]

$\Gamma \vdash \mathsf{if}\ u{<}v\ \mathsf{then}\ s\ \mathsf{else}\ t = t_2 \in T$ [Ax]
by <u>lessRedF</u>
$\Gamma \vdash t = t_2 \in T$ [Ax]
$\Gamma \vdash u \geq v$ [Ax]

$\Gamma \vdash 0 \leq s\ \mathsf{rem}\ t \ \wedge\ s\ \mathsf{rem}\ t < t$ [Ax]
by <u>remBounds1</u>
$\Gamma \vdash 0 \leq s$ [Ax]
$\Gamma \vdash 0 < t$ [Ax]

$\Gamma \vdash 0 \leq s\ \mathsf{rem}\ t \ \wedge\ s\ \mathsf{rem}\ t < -t$ [Ax]
by <u>remBounds2</u>
$\Gamma \vdash 0 \leq s$ [Ax]
$\Gamma \vdash t < 0$ [Ax]

$\Gamma \vdash s\ \mathsf{rem}\ t \leq 0 \ \wedge\ s\ \mathsf{rem}\ t > -t$ [Ax]
by <u>remBounds3</u>
$\Gamma \vdash s \leq 0$ [Ax]
$\Gamma \vdash 0 < t$ [Ax]

$\Gamma \vdash s\ \mathsf{rem}\ t \leq 0 \ \wedge\ s\ \mathsf{rem}\ t > t$ [Ax]
by <u>remBounds4</u>
$\Gamma \vdash s \leq 0$ [Ax]
$\Gamma \vdash t < 0$ [Ax]

$\Gamma \vdash s = (s \div t) * t + (s\ \mathsf{rem}\ t)$ [Ax]
by <u>divremSum</u>
$\Gamma \vdash s \in \mathbb{Z}$ [Ax]
$\Gamma \vdash t \neq 0$ [Ax]

$\Gamma \vdash s_1{<}t_1 = s_2{<}t_2 \in \mathsf{U}_j$ [Ax]
by <u>ltEq</u>
$\Gamma \vdash s_1 = s_2$ [Ax]
$\Gamma \vdash t_1 = t_2$ [Ax]

$\Gamma \vdash \mathsf{Axiom} \in s{<}t$ [Ax]
by <u>axiomEq_lt</u>
$\Gamma \vdash s{<}t$ [Ax]

A.5.2 Void

$\Gamma \vdash \mathsf{void} \in \mathsf{U}_j$ [Ax]
by <u>voidEq</u>

$\Gamma \vdash \mathsf{any}(s) = \mathsf{any}(t) \in T$ [Ax]
by <u>anyEq</u>
$\Gamma \vdash s = t \in \mathsf{void}$ [Ax]

$\Gamma,\ z{:}\,\mathsf{void},\ \Delta \vdash C$ [**ext** $\mathsf{any}(z)$]
by <u>voidE i</u>

A.5.3 Universen

$\Gamma \vdash \mathsf{U}_j \in \mathsf{U}_k$ [Ax]
by <u>univEq *</u>

$\Gamma \vdash T \in \mathsf{U}_k$ [Ax]
by <u>cumulativity j *</u>
$\Gamma \vdash T \in \mathsf{U}_j$ [Ax]

A.5.4 Atom

$\Gamma \vdash \mathsf{Atom} = \mathsf{Atom} \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$
by `atomEq`

$\Gamma \vdash \text{"}string\text{"} \in \mathsf{Atom} \;\lfloor\mathsf{Ax}\rfloor$
by `tokEq`

$\Gamma \vdash \mathsf{Atom} \;\lfloor\textbf{ext } \text{"}string\text{"}\rfloor$
by `tokI` $\text{"}string\text{"}$

$\Gamma \vdash \mathsf{if}\ u_1{=}v_1\ \mathsf{then}\ s_1\ \mathsf{else}\ t_1$
$= \mathsf{if}\ u_2{=}v_2\ \mathsf{then}\ s_2\ \mathsf{else}\ t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
by `atom_eqEq`
$\Gamma \vdash u_1{=}u_2 \in \mathsf{Atom} \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash v_1{=}v_2 \in \mathsf{Atom} \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma,\ v{:}\ u_1{=}v_1 \in \mathsf{Atom} \vdash s_1 = s_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma,\ v{:}\ \neg(u_1{=}v_1 \in \mathsf{Atom}) \vdash t_1 = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma \vdash \mathsf{if}\ u{=}v\ \mathsf{then}\ s\ \mathsf{else}\ t = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
by `atom_eqRedT`
$\Gamma \vdash s = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash u{=}v \in \mathsf{Atom} \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma \vdash \mathsf{if}\ u{=}v\ \mathsf{then}\ s\ \mathsf{else}\ t = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
by `atom_eqRedF`
$\Gamma \vdash t = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash \neg(u{=}v \in \mathsf{Atom}) \;\lfloor\mathsf{Ax}\rfloor$

A.5.5 Funktionenraum

$\Gamma \vdash x_1{:}S_1{\rightarrow}T_1 = x_2{:}S_2{\rightarrow}T_2 \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$
by `functionEq`
$\Gamma \vdash S_1 = S_2 \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma,\ x{:}S_1 \vdash T_1[x/x_1] = T_2[x/x_2] \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma \vdash \lambda x_1.t_1 = \lambda x_2.t_2 \in x{:}S{\rightarrow}T \;\lfloor\mathsf{Ax}\rfloor$
by `lambdaEq` j
$\Gamma,\ x'{:}S \vdash t_1[x'/x_1] = t_2[x'/x_2] \in T[x'/x] \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash S \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma \vdash x{:}S{\rightarrow}T \;\lfloor\textbf{ext } \lambda x'.t\rfloor$
by `lambdaI` j
$\Gamma,\ x'{:}S \vdash T[x'/x] \;\lfloor\textbf{ext } t\rfloor$
$\Gamma \vdash S \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma \vdash f_1\,t_1 = f_2\,t_2 \in T[t_1/x] \;\lfloor\mathsf{Ax}\rfloor$
by `applyEq` $x{:}S{\rightarrow}T$
$\Gamma \vdash f_1 = f_2 \in x{:}S{\rightarrow}T \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash t_1 = t_2 \in S \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma,\ f{:}\ x{:}S{\rightarrow}T,\ \Delta \vdash C \;\lfloor\textbf{ext } t[f\,s, \mathsf{Axiom}\,/\,y, v]\rfloor$
by `functionE` $i\ s$
$\Gamma,\ f{:}\ x{:}S{\rightarrow}T,\ \Delta \vdash s \in S \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma,\ f{:}\ x{:}S{\rightarrow}T,\ y{:}T[s/x],$
$v{:}\ y = f\,s \in T[s/x],\ \Delta \vdash C \;\lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash (\lambda x.t)\,s = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$
by `applyRed`
$\Gamma \vdash t[s/x] = t_2 \in T \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma \vdash f_1 = f_2 \in x{:}S{\rightarrow}T \;\lfloor\textbf{ext } t\rfloor$
by `functionExt` $j\ x_1{:}S_1{\rightarrow}T_1\ x_2{:}S_2{\rightarrow}T_2$
$\Gamma,\ x'{:}S \vdash f_1\,x' = f_2\,x' \in T[x'/x] \;\lfloor\textbf{ext } t\rfloor$
$\Gamma \vdash S \in \mathsf{U}_j \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash f_1 \in x_1{:}S_1{\rightarrow}T_1 \;\lfloor\mathsf{Ax}\rfloor$
$\Gamma \vdash f_2 \in x_2{:}S_2{\rightarrow}T_2 \;\lfloor\mathsf{Ax}\rfloor$

$\Gamma,\ f{:}\ S{\rightarrow}T,\ \Delta \vdash C \;\lfloor\textbf{ext } t[f\,s,/y]\rfloor$
by `functionE_indep` i
$\Gamma,\ f{:}\ S{\rightarrow}T,\ \Delta \vdash S \;\lfloor\textbf{ext } s\rfloor$
$\Gamma,\ f{:}\ S{\rightarrow}T,\ y{:}T,\ \Delta \vdash C \;\lfloor\textbf{ext } t\rfloor$

A.5.6 Produktraum

$\Gamma \vdash x_1{:}S_1{\times}T_1 = x_2{:}S_2{\times}T_2 \in \mathsf{U}_j$ ⌊Ax⌋
by productEq
  $\Gamma \vdash S_1 = S_2 \in \mathsf{U}_j$ ⌊Ax⌋
  $\Gamma,\ x'{:}S_1 \vdash T_1[x'/x_1] = T_2[x'/x_2] \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash \langle s_1, t_1\rangle = \langle s_2, t_2\rangle \in x{:}S{\times}T$ ⌊Ax⌋
by pairEq j
  $\Gamma \vdash s_1 = s_2 \in S$ ⌊Ax⌋
  $\Gamma \vdash t_1 = t_2 \in T[s_1/x]$ ⌊Ax⌋
  $\Gamma,\ x'{:}S \vdash T[x'/x] \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash x{:}S{\times}T$ ⌊**ext** $\langle s, t\rangle$⌋
by pairI j s
  $\Gamma \vdash s \in S$ ⌊Ax⌋
  $\Gamma \vdash T[s/x]$ ⌊**ext** t⌋
  $\Gamma,\ x'{:}S \vdash T[x'/x] \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash$ let $\langle x_1, y_1\rangle = e_1$ in t_1
  = let $\langle x_2, y_2\rangle = e_2$ in $t_2 \in C[e_1/z]$ ⌊Ax⌋
by spreadEq z C $x{:}S{\times}T$
  $\Gamma \vdash e_1 = e_2 \in x{:}S{\times}T$ ⌊Ax⌋
  $\Gamma,\ s{:}S,\ t{:}T[s/x],\ y{:}\ e_1{=}\langle s, t\rangle \in x{:}S{\times}T$
    $\vdash t_1[s,t/x_1,y_1] = t_2[s,t/x_2,y_2] \in C[\langle s, t\rangle/z]$ ⌊Ax⌋

$\Gamma,\ z{:}\ x{:}S{\times}T,\ \Delta \vdash C$ ⌊**ext** let $\langle s, t\rangle = z$ in u⌋
by productE i
  $\Gamma,\ z{:}\ x{:}S{\times}T,\ s{:}S,\ t{:}T[s/x],\ \Delta[\langle s, t\rangle/z]$
    $\vdash C[\langle s, t\rangle/z]$ ⌊**ext** u⌋

$\Gamma \vdash$ let $\langle x, y\rangle = \langle s, t\rangle$ in $u = t_2 \in T$ ⌊Ax⌋
by spreadRed
  $\Gamma \vdash u[s,t/x,y] = t_2 \in T$ ⌊Ax⌋

$\Gamma \vdash \langle s_1, t_1\rangle = \langle s_2, t_2\rangle \in S{\times}T$ ⌊Ax⌋
by pairEq_indep
  $\Gamma \vdash s_1 = s_2 \in S$ ⌊Ax⌋
  $\Gamma \vdash t_1 = t_2 \in T$ ⌊Ax⌋

$\Gamma \vdash S{\times}T$ ⌊**ext** $\langle s, t\rangle$⌋
by pairI_indep
  $\Gamma \vdash S$ ⌊**ext** s⌋
  $\Gamma \vdash T$ ⌊**ext** t⌋

A.5.7 Disjunkte Vereinigung (Summe)

$\Gamma \vdash S_1{+}T_1 = S_2{+}T_2 \in \mathsf{U}_j$ ⌊Ax⌋
by unionEq
  $\Gamma \vdash S_1 = S_2 \in \mathsf{U}_j$ ⌊Ax⌋
  $\Gamma \vdash T_1 = T_2 \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash \mathsf{inl}(s_1) = \mathsf{inl}(s_2) \in S{+}T$ ⌊Ax⌋
by inlEq j
  $\Gamma \vdash s_1 = s_2 \in S$ ⌊Ax⌋
  $\Gamma \vdash T \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash S{+}T$ ⌊**ext** $\mathsf{inl}(s)$⌋
by inlI j
  $\Gamma \vdash S$ ⌊**ext** s⌋
  $\Gamma \vdash T \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash \mathsf{inr}(t_1) = \mathsf{inr}(t_2) \in S{+}T$ ⌊Ax⌋
by inrEq j
  $\Gamma \vdash t_1 = t_2 \in T$ ⌊Ax⌋
  $\Gamma \vdash S \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash S{+}T$ ⌊**ext** $\mathsf{inr}(t)$⌋
by inrI j
  $\Gamma \vdash T$ ⌊**ext** t⌋
  $\Gamma \vdash S \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash \mathsf{case}\ e_1\ \mathsf{of}\ \mathsf{inl}(x_1) \mapsto u_1 \mid \mathsf{inr}(y_1) \mapsto v_1$
  $= \mathsf{case}\ e_2\ \mathsf{of}\ \mathsf{inl}(x_2) \mapsto u_2 \mid \mathsf{inr}(y_2) \mapsto v_2$
  $\in C[e_1/z]$ ⌊Ax⌋
by decideEq z C $S{+}T$
  $\Gamma \vdash e_1 = e_2 \in S{+}T$ ⌊Ax⌋
  $\Gamma,\ s{:}S,\ y{:}\ e_1{=}\mathsf{inl}(s) \in S{+}T$
    $\vdash u_1[s/x_1] = u_2[s/x_2] \in C[\mathsf{inl}(s)/z]$ ⌊Ax⌋
  $\Gamma,\ t{:}T,\ y{:}\ e_1{=}\mathsf{inr}(t) \in S{+}T$
    $\vdash v_1[t/y_1] = v_2[t/y_2] \in C[\mathsf{inr}(t)/z]$ ⌊Ax⌋

$\Gamma,\ z{:}S{+}T,\ \Delta \vdash C$
  ⌊**ext** $\mathsf{case}\ z\ \mathsf{of}\ \mathsf{inl}(x) \mapsto u \mid \mathsf{inr}(y) \mapsto v$⌋
by unionE i
  $\Gamma,\ z{:}S{+}T,\ x{:}S,\ \Delta[\mathsf{inl}(x)/z]$
    $\vdash C[\mathsf{inl}(x)/z]$ ⌊**ext** u⌋
  $\Gamma,\ z{:}S{+}T,\ y{:}T,\ \Delta[\mathsf{inr}(y)/z]$
    $\vdash C[\mathsf{inr}(y)/z]$ ⌊**ext** v⌋

$\Gamma \vdash \mathsf{case}\ \mathsf{inl}(s)\ \mathsf{of}\ \mathsf{inl}(x) \mapsto u \mid \mathsf{inr}(y) \mapsto v$
  $= t_2 \in T$ ⌊Ax⌋
by decideRedL
  $\Gamma \vdash u[s/x] = t_2 \in T$ ⌊Ax⌋

$\Gamma \vdash \mathsf{case}\ \mathsf{inr}(t)\ \mathsf{of}\ \mathsf{inl}(x) \mapsto u \mid \mathsf{inr}(y) \mapsto v$
  $= t_2 \in T$ ⌊Ax⌋
by decideRedR
  $\Gamma \vdash v[t/y] = t_2 \in T$ ⌊Ax⌋

A.5.8 Gleichheit

$\Gamma \vdash s_1 = t_1 \in T_1 \;=\; s_2 = t_2 \in T_2 \;\in\; \mathsf{U}_j$ ⌊Ax⌋
by equalityEq
$\Gamma \vdash T_1 = T_2 \in \mathsf{U}_j$ ⌊Ax⌋
$\Gamma \vdash s_1 = s_2 \in T_1$ ⌊Ax⌋
$\Gamma \vdash t_1 = t_2 \in T_1$ ⌊Ax⌋

$\Gamma \vdash \mathsf{Axiom} \in s = t \in T$ ⌊Ax⌋
by axiomEq
$\Gamma \vdash s = t \in T$ ⌊Ax⌋

$\Gamma,\ z{:}\ s = t \in T,\ \Delta \vdash C$ ⌊**ext** u⌋
by equalityE i
$\Gamma,\ z{:}\ s = t \in T,\ \Delta[\mathsf{Axiom}/z]$
$\vdash C[\mathsf{Axiom}/z]$ ⌊**ext** u⌋

$\Gamma,\ x{:}T,\ \Delta \vdash x \in T$ ⌊Ax⌋
by hypEq i

$\Gamma \vdash s = t \in T$ ⌊Ax⌋
by symmetry
$\Gamma \vdash t = s \in T$ ⌊Ax⌋

$\Gamma \vdash s = t \in T$ ⌊Ax⌋
by transitivity t'
$\Gamma \vdash s = t' \in T$ ⌊Ax⌋
$\Gamma \vdash t' = t \in T$ ⌊Ax⌋

$\Gamma \vdash C[s/x]$ ⌊**ext** u⌋
by subst $j\ \ s = t \in T\ \ x\ \ C$
$\Gamma \vdash s = t \in T$ ⌊Ax⌋
$\Gamma \vdash C[t/x]$ ⌊**ext** u⌋
$\Gamma,\ x{:}T \vdash C \in \mathsf{U}_j$ ⌊Ax⌋

A.5.9 Listen

$\Gamma \vdash T_1\,\mathsf{list} = T_2\,\mathsf{list} \in \mathsf{U}_j$ ⌊Ax⌋
by listEq
$\Gamma \vdash T_1 = T_2 \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash [\,] \in T\,\mathsf{list}$ ⌊Ax⌋
by nilEq j
$\Gamma \vdash T \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash T\,\mathsf{list}$ ⌊**ext** [ ]⌋
by nilI j
$\Gamma \vdash T \in \mathsf{U}_j$ ⌊Ax⌋

$\Gamma \vdash t_1 . l_1 = t_2 . l_2 \in T\,\mathsf{list}$ ⌊Ax⌋
by consEq
$\Gamma \vdash t_1 = t_2 \in T$ ⌊Ax⌋
$\Gamma \vdash l_1 = l_2 \in T\,\mathsf{list}$ ⌊Ax⌋

$\Gamma \vdash T\,\mathsf{list}$ ⌊**ext** $t.l$⌋
by consI
$\Gamma \vdash T$ ⌊**ext** t⌋
$\Gamma \vdash T\,\mathsf{list}$ ⌊**ext** l⌋

$\Gamma \vdash \mathsf{list\_ind}(s_1;\ base_1;\ x_1, l_1, f_{xl1} . t_1)$
$= \mathsf{list\_ind}(s_2;\ base_2;\ x_2, l_2, f_{xl2} . t_2) \in T[s_1/z]$ ⌊Ax⌋
by list_indEq $z\ \ T\ \ S\,\mathsf{list}$
$\Gamma \vdash s_1 = s_2 \in S\,\mathsf{list}$ ⌊Ax⌋
$\Gamma \vdash base_1 = base_2 \in T[[\,]/z]$ ⌊Ax⌋
$\Gamma,\ x{:}S,\ l{:}S\,\mathsf{list},\ f_{xl}{:}T[l/z] \vdash$
$t_1[x, l, f_{xl} / x_1, l_1, f_{xl1}] = t_1[x, l, f_{xl} / x_2, l_2, f_{xl2}]$
$\in T[x.l/z]$ ⌊Ax⌋

$\Gamma,\ z{:}\ T\,\mathsf{list},\ \Delta \vdash C$
⌊**ext** $\mathsf{list\_ind}(z;\ base;\ x, l, f_{xl} . t)$⌋
by listE i
$\Gamma,\ z{:}\ T\,\mathsf{list},\ \Delta \vdash C[[\,]/z]$ ⌊**ext** $base$⌋
$\Gamma,\ z{:}\ T\,\mathsf{list},\ \Delta,\ x{:}T,\ l{:}T\,\mathsf{list},\ f_{xl}{:}C[l/z]$
$\vdash C[x.l/z]$ ⌊**ext** t⌋

$\Gamma \vdash \mathsf{list\_ind}([\,];\ base;\ x, l, f_{xl} . t) = t_2 \in T$ ⌊Ax⌋
by list_indRedBase
$\Gamma \vdash base = t_2 \in T$ ⌊Ax⌋

$\Gamma \vdash \mathsf{list\_ind}(s.u;\ base;\ x, l, f_{xl} . t) = t_2 \in T$ ⌊Ax⌋
by list_indRedUp
$\Gamma \vdash t[s, u, \mathsf{list\_ind}(u;\ base;\ x, l, f_{xl} . t)\ /x, l, f_{xl}]$
$= t_2 \in T$ ⌊Ax⌋

A.5.10 Teilmengen

$\Gamma \vdash \{x_1{:}S_1 \mid T_1\} = \{x_2{:}S_2 \mid T_2\} \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
by setEq
$\quad\Gamma \vdash S_1 = S_2 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x{:}S_1 \vdash T_1[x/x_1] = T_2[x/x_2] \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$

$\Gamma \vdash s = t \in \{x{:}S \mid T\}$ $\lfloor$Ax$\rfloor$
by elementEq j
$\quad\Gamma \vdash s = t \in S$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash T[s/x]$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x'{:}S \vdash T[x'/x] \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$

$\Gamma \vdash \{x{:}S \mid T\}$ $\lfloor$**ext** $s\rfloor$
by elementI j s
$\quad\Gamma \vdash s \in S$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash T[s/x]$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x'{:}S \vdash T[x'/x] \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$

$\Gamma,\ z{:}\{x{:}S \mid T\},\ \Delta \vdash C$ $\lfloor$**ext** $(\lambda y.t)\ z\rfloor$
by setE i
$\quad\Gamma,\ z{:}\{x{:}S \mid T\},\ y{:}S,\ [\![v]\!]{:}T[y/x],\ \Delta[y/z]$
$\qquad\vdash C[y/z]$ $\lfloor$**ext** $t\rfloor$

$\Gamma \vdash s = t \in \{S \mid T\}$ $\lfloor$Ax$\rfloor$
by elementEq_indep
$\quad\Gamma \vdash s = t \in S$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash T$ $\lfloor$Ax$\rfloor$

$\Gamma \vdash \{S \mid T\}$ $\lfloor$**ext** $s\rfloor$
by elementI_indep
$\quad\Gamma \vdash S$ $\lfloor$**ext** $s\rfloor$
$\quad\Gamma \vdash T$ $\lfloor$Ax$\rfloor$

A.5.11 Quotienten

$\Gamma \vdash x_1,y_1 : T_1/\!/E_1 = x_2,y_2 : T_2/\!/E_2 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
by quotientEq_weak
$\quad\Gamma \vdash T_1 = T_2 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x{:}T_1,\ y{:}T_1$
$\qquad\vdash E_1[x,y/x_1,y_1] = E_2[x,y/x_2,y_2] \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x{:}T_1,\ y{:}T_1 \vdash E_1[x,x/x_1,y_1]$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x{:}T_1,\ y{:}T_1,\ v{:}\ E_1[x,y/x_1,y_1]$
$\qquad\vdash E_1[y,x/x_1,y_1]$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ x{:}T_1,\ y{:}T_1,\ z{:}T_1,\ v{:}\ E_1[x,y/x_1,y_1],$
$\qquad v'{:}\ E_1[y,z/x_1,y_1] \vdash E_1[x,z/x_1,y_1]$ $\lfloor$Ax$\rfloor$

$\Gamma \vdash x_1,y_1 : T_1/\!/E_1 = x_2,y_2 : T_2/\!/E_2 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
by quotientEq
$\quad\Gamma \vdash x_1,y_1 : T_1/\!/E_1 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash x_2,y_2 : T_2/\!/E_2 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash T_1 = T_2 \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ v{:}\ T_1 = T_2 \in \mathsf{U}_j,\ x{:}T_1,\ y{:}T_1$
$\qquad\vdash E_1[x,y/x_1,y_1] \Rightarrow E_2[x,y/x_2,y_2]$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ v{:}\ T_1 = T_2 \in \mathsf{U}_j,\ x{:}T_1,\ y{:}T_1$
$\qquad\vdash E_2[x,y/x_2,y_2] \Rightarrow E_1[x,y/x_1,y_1]$ $\lfloor$Ax$\rfloor$

$\Gamma \vdash s = t \in x,y : T/\!/E$ $\lfloor$Ax$\rfloor$
by memberEq_weak j
$\quad\Gamma \vdash x,y : T/\!/E \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash s = t \in T$ $\lfloor$Ax$\rfloor$

$\Gamma \vdash x,y : T/\!/E$ $\lfloor$**ext** $t\rfloor$
by memberI j
$\quad\Gamma \vdash x,y : T/\!/E \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash T$ $\lfloor$**ext** $t\rfloor$

$\Gamma \vdash s = t \in x,y : T/\!/E$ $\lfloor$Ax$\rfloor$
by memberEq j
$\quad\Gamma \vdash x,y : T/\!/E \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash s \in T$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash t \in T$ $\lfloor$Ax$\rfloor$
$\quad\Gamma \vdash E[s,t/x,y]$ $\lfloor$Ax$\rfloor$

$\Gamma,\ v{:}\ s = t \in x,y : T/\!/E,\ \Delta \vdash C$ $\lfloor$**ext** $u\rfloor$
by quotient_eqE i j
$\quad\Gamma,\ v{:}\ s = t \in x,y : T/\!/E,\ [\![v']\!]{:}E[s,t/x,y],\ \Delta$
$\qquad\vdash C$ $\lfloor$**ext** $u\rfloor$
$\quad\Gamma,\ v{:}\ s = t \in x,y : T/\!/E,\ \Delta,\ x'{:}T,\ y'{:}T$

$\Gamma,\ z{:}\ x,y : T/\!/E,\ \Delta \vdash s = t \in S$ $\lfloor$Ax$\rfloor$
by quotientE i j
$\quad\Gamma,\ z{:}\ x,y : T/\!/E,\ \Delta,\ x'{:}T,\ y'{:}T$
$\qquad\vdash E[x',y'/x,y] \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$
$\quad\Gamma,\ z{:}\ x,y : T/\!/E,\ \Delta \vdash S \in \mathsf{U}_j$ $\lfloor$Ax$\rfloor$

A.5.12 Induktive Typen

$\Gamma \vdash \text{rectype } X_1 = T_{X1} = \text{rectype } X_2 = T_{X2} \in U_j \quad \lfloor \text{Ax} \rfloor$
by recEq
$\Gamma,\ X{:}U_j \vdash T_{X1}[X/X_1] = T_{X2}[X/X_2] \in U_j \quad \lfloor \text{Ax} \rfloor$

$\Gamma \vdash s = t \in \text{rectype } X = T_X \quad \lfloor \text{Ax} \rfloor$
by rec_memEq j
$\Gamma \vdash s = t \in T_X[\text{rectype } X = T_X / X] \quad \lfloor \text{Ax} \rfloor$
$\Gamma \vdash \text{rectype } X = T_X \in U_j \quad \lfloor \text{Ax} \rfloor$

$\Gamma \vdash \text{let}^* f_1(x_1) = t_1 \text{ in } f_1(e_1)$
$= \text{let}^* f_2(x_2) = t_2 \text{ in } f_2(e_2) \in T[e_1/z] \quad \lfloor \text{Ax} \rfloor$
by rec_indEq $z\ T$ rectype $X = T_X\ j$
$\Gamma \vdash e_1 = e_2 \in \text{rectype } X = T_X \quad \lfloor \text{Ax} \rfloor$
$\Gamma \vdash \text{rectype } X = T_X \in U_j \quad \lfloor \text{Ax} \rfloor$
$\Gamma,\ P{:}\ (\text{rectype } X = T_X) \rightarrow \mathbb{P}_j,$
$f{:}\ (y{:}\{x{:}\text{rectype } X = T_X \mid P(x)\} \rightarrow T[y/z]),$
$x{:}\ T_X[\{x{:}\text{rectype } X = T_X \mid P(x)\} / X]$
$\vdash t_1[f, x / f_1, x_1] = t_2[f, x / f_2, x_2] \in T[x/z] \quad \lfloor \text{Ax} \rfloor$

$\Gamma \vdash \text{rectype } X = T_X \quad \lfloor \textbf{ext } t \rfloor$
by rec_memI j
$\Gamma \vdash T_X[\text{rectype } X = T_X / X] \quad \lfloor \textbf{ext } t \rfloor$
$\Gamma \vdash \text{rectype } X = T_X \in U_j \quad \lfloor \text{Ax} \rfloor$

$\Gamma,\ z{:}\ \text{rectype } X = T_X,\ \Delta \vdash C$
$\lfloor \textbf{ext } \text{let}^* f(x) = t[\lambda y.\Lambda / P] \text{ in } f(z) \rfloor$
by recE $i\ j$
$\Gamma,\ z{:}\ \text{rectype } X = T_X,\ \Delta$
$\vdash \text{rectype } X = T_X \in U_j \quad \lfloor \text{Ax} \rfloor$
$\Gamma,\ z{:}\ \text{rectype } X = T_X,\ \Delta$
$P{:}\ (\text{rectype } X = T_X) \rightarrow \mathbb{P}_j,$
$f{:}\ (y{:}\{x{:}\text{rectype } X = T_X \mid P(x)\} \rightarrow C[y/z]),$
$x{:}\ T_X[\{x{:}\text{rectype } X = T_X \mid P(x)\} / X]$
$\vdash C[x/z] \quad \lfloor \textbf{ext } t \rfloor$

$\Gamma,\ z{:}\ \text{rectype } X = T_X,\ \Delta \vdash C \quad \lfloor \textbf{ext } t[z/x] \rfloor$
by recE_unroll i
$\Gamma,\ z{:}\ \text{rectype } X = T_X,\ \Delta,$
$x{:}\ T_X[\text{rectype } X = T_X / X],$
$v{:}\ z{=}x \in T_X[\text{rectype } X = T_X / X] \vdash C[x/z] \quad \lfloor \textbf{ext } t \rfloor$

A.5.13 Rekursive Funktionen

$\Gamma \vdash S_1 \not\rightarrow T_1 = S_2 \not\rightarrow T_2 \in U_j \quad \lfloor \text{Ax} \rfloor$
by pfunEq
$\Gamma \vdash S_1 = S_2 \in U_j \quad \lfloor \text{Ax} \rfloor$
$\Gamma \vdash T_1 = T_2 \in U_j \quad \lfloor \text{Ax} \rfloor$

$\Gamma \vdash (\text{letrec } f_1(x_1) = t_1)$
$= (\text{letrec } f_2(x_2) = t_2) \in S \not\rightarrow T \quad \lfloor \text{Ax} \rfloor$
by fixEq j
$\Gamma \vdash \text{letrec } f_1(x_1) = t_1 \in S \not\rightarrow T \quad \lfloor \text{Ax} \rfloor$
$\Gamma \vdash \text{letrec } f_2(x_2) = t_2 \in S \not\rightarrow T \quad \lfloor \text{Ax} \rfloor$
$\Gamma \vdash \{x{:}S \mid \text{dom}(\text{letrec } f_1(x_1) = t_1)(x)\}$
$= \{x{:}S \mid \text{dom}(\text{letrec } f_2(x_2) = t_2)(x)\} \in U_j \quad \lfloor \text{Ax} \rfloor$
$\Gamma,\ y{:}\ \{x{:}S \mid \text{dom}(\text{letrec } f_1(x_1) = t_1)(x)\}$
$\vdash (\text{letrec } f_1(x_1) = t_1)\ (y)$
$= (\text{letrec } f_2(x_2) = t_2)\ (y) \in T \quad \lfloor \text{Ax} \rfloor$

$\Gamma \vdash \text{letrec } f(x) = t \in S \not\rightarrow T \quad \lfloor \text{Ax} \rfloor$
by fixMem j
$\Gamma \vdash S \not\rightarrow T \in U_j \quad \lfloor \text{Ax} \rfloor$
$\Gamma,\ f'{:}S \not\rightarrow T,\ x'{:}S \vdash \mathcal{E}[\![t[f', x' / f, x]]\!] \in U_j \quad \lfloor \text{Ax} \rfloor$
$\Gamma,\ f'{:}S \not\rightarrow T,\ x'{:}S,\ \mathcal{E}[\![t[f', x' / f, x]]\!]$
$\vdash t[f', x' / f, x] \in T \quad \lfloor \text{Ax} \rfloor$

$\Gamma \vdash f_1\ (t_1) = f_2\ (t_2) \in T \quad \lfloor \text{Ax} \rfloor$
by apply_pEq $S \not\rightarrow T$
$\Gamma \vdash f_1 = f_2 \in S \not\rightarrow T \quad \lfloor \text{Ax} \rfloor$
$\Gamma \vdash t_1 = t_2 \in \{x{:}S \mid \text{dom}(f_1)(x)\} \quad \lfloor \text{Ax} \rfloor$

$\Gamma,\ f{:}\ S \not\rightarrow T,\ \Delta \vdash C \quad \lfloor \textbf{ext } t[f\ (s), \text{Axiom} / y, v] \rfloor$
by pfunE $i\ s$
$\Gamma,\ f{:}\ S \not\rightarrow T,\ \Delta \vdash s \in \{x{:}S \mid \text{dom}(f_1)(x)\} \quad \lfloor \text{Ax} \rfloor$
$\Gamma,\ f{:}\ S \not\rightarrow T,\ \Delta,\ y{:}T,\ v{:}\ y{=}f\ (s) \in T \vdash C \quad \lfloor \textbf{ext } t \rfloor$

$\Gamma \vdash (\text{letrec } f(x) = t)\ (u) = t_2 \in T \quad \lfloor \text{Ax} \rfloor$
by apply_pRed

$\Gamma \vdash \text{dom}(f_1) = \text{dom}(f_2) \in S \rightarrow \mathbb{P}_j \quad \lfloor \text{Ax} \rfloor$
by domEq $S \not\rightarrow T$

A.5.14 Strukturelle Regeln und sonstige Regeln

$\Gamma,\ x{:}T,\ \Delta \vdash T \ \lfloor\textbf{ext } x\rfloor$
by `hypothesis` i

$\Gamma \vdash T \ \lfloor\textbf{ext } t\rfloor$
by `intro` t
$\Gamma \vdash t \in T \ \lfloor\text{Ax}\rfloor$

$\Gamma,\Delta \vdash C \ \lfloor\textbf{ext } (\lambda x.t)\ s\rfloor$
by `cut` $i\ T$
$\Gamma,\Delta \vdash T \ \lfloor\textbf{ext } s\rfloor$
$\Gamma,x{:}T,\Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma,\ x{:}T,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `thin` i
$\Gamma,\Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `compute` $tagC$
$\Gamma \vdash C{\downarrow}_{tagC} \ \lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `rev_compute` $tagC$
$\Gamma, \vdash C{\uparrow}_{tagC} \ \lfloor\textbf{ext } t\rfloor$

$\Gamma,\ z{:}\,T,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `computeHyp` $i\ tagT$
$\Gamma,\ z{:}\,T{\downarrow}_{tagT},\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma,\ z{:}\,T,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `rev_computeHyp` $i\ tagT$
$\Gamma,\ z{:}\,T{\uparrow}_{tagT},\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `unfold` *def-name*
$\Gamma \vdash C_{\downarrow} \ \lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash C_{\downarrow} \ \lfloor\textbf{ext } t\rfloor$
by `fold` *def-name*
$\Gamma \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma,\ z{:}\,T,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `unfoldHyp` i *def-name*
$\Gamma,\ z{:}\,T_{\downarrow},\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma,\ z{:}\,T_{\downarrow},\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `foldHyp` i *def-name*
$\Gamma,\ z{:}\,T,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$

$\Gamma,\ z{:}\,T,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `replaceHyp` $i\ S\ j$
$\Gamma,\ z{:}\,S,\ \Delta \vdash C \ \lfloor\textbf{ext } t\rfloor$
$\Gamma,\ z{:}\,T,\ \Delta \vdash T = S \in \mathsf{U}_j \ \lfloor\text{Ax}\rfloor$

$\Gamma \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `lemma` *theorem-name*

$\Gamma \vdash t \in T \ \lfloor\text{Ax}\rfloor$
by `extract` *theorem-name*

$\Gamma \vdash C \ \lfloor\textbf{ext } t[\sigma]\rfloor$
by `instantiate` $\Gamma'\ C'\ \sigma$
$\Gamma' \vdash C' \ \lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash C \ \lfloor\textbf{ext } t[y/x]\rfloor$
by `rename` $y\ x$
$\Gamma[x/y] \vdash C[x/y] \ \lfloor\textbf{ext } t\rfloor$

$\Gamma \vdash s = t \in T \ \lfloor\text{Ax}\rfloor$
by `equality`

$\Gamma \vdash C \ \lfloor\textbf{ext } t\rfloor$
by `arith` j
$\Gamma \vdash s_1 \in \mathbb{Z} \ \lfloor\text{Ax}\rfloor$
$\vdots$