On Presenting Monotonicity and On EA $\Rightarrow$ AE

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Abstract

Two independent topics are treated. First, the problem of weakening/strengthening steps is discussed and a form for substantiating such steps is proposed. Second, a simple proof of $(\exists x \mid R.x : (\forall y \mid S.y : P.x.y)) \Rightarrow (\forall y \mid S.y : (\exists x \mid R.x : P.x.y))$ is presented, which uses the idea of a witness for an existential quantification.

Introduction to Monotonicity

A function f is *monotonic* in its argument if $x \Rightarrow y$ implies $f.x \Rightarrow f.y$ (for all x, y). It is *antimonotonic* if $x \Rightarrow y$ implies $f.x \Leftarrow f.y$.

We propose a notation for signalling the use of monotonicity and antimonotonicity in calculational proofs. In weakening/strengthening steps that use monotonicity/antimonotonicity together with a theorem like $P \Rightarrow P \vee Q$, we propose using the hints given in the following two examples.

$$
\begin{array}{ll}
 & (\forall x \mid : P \wedge R) \\
\Rightarrow & \quad \langle \text{Monotonicity: Weakening } P \Rightarrow P \vee Q \rangle \\
 & (\forall x \mid : (P \vee Q) \wedge R)
\end{array}
$$

$$
\begin{array}{ll}
 & \neg(\forall x \mid \neg P \wedge R : S) \\
\Leftarrow & \quad \langle \text{Antimonotonicity: Weakening } P \Rightarrow P \vee Q \rangle \\
 & \neg(\forall x \mid \neg(P \vee Q) \wedge R : S)
\end{array}
$$

The rest of this note presents a (well-known) theorem concerning monotonicity and explains the reason for introducing the new notation in hints.

Monotonicity properties of the logical operators

It is known that $\vee$ and $\wedge$ are monotonic in each of their operands, that negation is antimonotonic in its operand, that $\Rightarrow$ is monotonic in its consequent, and that $\Rightarrow$ is antimonotonic in its antecedent. Also, $(\forall x \mid R : P)$ is monotonic in P but antimonotonic in R, while $(\exists x \mid R : P)$ is monotonic in both P and R. Formally, we have:

(1) **Monotonic $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(2) **Monotonic $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

(3) **Antimonotonic $\neg$:** $(p \Rightarrow q) \Rightarrow (\neg p \Leftarrow \neg q)$

(4) **Monotonic consequent:** $(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$

(5) **Antimonotonic antecedent:** $(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Leftarrow (q \Rightarrow r))$

(6) **Monotonic $\forall$-body:** $(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\forall x \mid R : P) \Rightarrow (\forall x \mid R : Q))$

(7) **Antimonotonic $\forall$-range:** $(\forall x \mid \neg R : P \Rightarrow Q) \Rightarrow ((\forall x \mid P : R) \Leftarrow (\forall x \mid Q : R))$

(8) **Monotonic $\exists$-body:** $(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\exists x \mid R : P) \Rightarrow (\exists x \mid R : Q))$

(9) **Monotonic $\exists$-range:** $(\forall x \mid R : P \Rightarrow Q) \Rightarrow ((\exists x \mid P : R) \Rightarrow (\exists x \mid Q : R))$

But, which of the following two formulas is valid, if either?

$$(\forall x \mid \neg P : S) \Rightarrow (\forall x \mid \neg(P \vee Q) : S)$$
$$(\forall x \mid \neg P : S) \Leftarrow (\forall x \mid \neg(P \vee Q) : S)$$

The answer is given by the following definition and theorem.

(10) **Definition.** Let z be a subformula of a formula E, where z is not within an operand of an equivalence (or an inequivalence). The position of z within E has *even parity* if it is nested within an even number of negations, antecedents, and ranges of universal quantifications; otherwise, it has *odd parity*.

(11) **Metatheorem Monotonicity.** Suppose $P \Rightarrow Q$ is a theorem. Let expression E contain exactly one occurrence of free variable z. Then:

(a) If the parity of the position of z in E is even,
$E[z := P] \Rightarrow E[z := Q]$ is a theorem.
(b) If the parity of the position of z in E is odd,
$E[z := P] \Leftarrow E[z := Q]$ is a theorem.

Sketch of proof. The proof is by induction on the structure of expression E. One can reduce the case analysis by first manipulating E so that one has only to deal with formulas that contain variables, constants, negations, disjunctions with z in the first operand, and existential quantifications with *true* ranges. Thus, make the following changes (in order).

- Replace $(\forall x \mid F1 : F2)$ by $\neg(\exists x \mid F1 : \neg F2)$.
- Replace $(\exists x \mid F1 : F2)$ by $(\exists x \mid : F1 \wedge F2)$.
- Replace $F1 \not\Leftarrow F2$ by $\neg(F1 \Leftarrow F2)$.

- Replace $F1 \Leftarrow F2$ by $F2 \Rightarrow F1$.
- Replace $F1 \not\Rightarrow F2$ by $\neg(F1 \Rightarrow F2)$.
- Replace $F1 \Rightarrow F2$ by $\neg F1 \vee F2$.
- Replace $F1 \wedge F2$ by $\neg(\neg F1 \vee \neg F2)$.
- If z is in the second operand $F2$ of $F1 \vee F2$, replace $F1 \vee F2$ by $F2 \vee F1$.

These manipulations do not change the parity of the position of z. Now, comes a straightforward proof by induction on the structure of the more restricted expressions E, which will rely on monotonic/antimonotonic properties (1), (3), and (8). □

Using Metatheorem Monotonicity

In a weakening/strengthening step of a calculation, the hint should explain why the step is sound Here is a simple example, where it is presumed that Weakening was proved earlier.

$$\begin{array}{ll} & P \\ \Rightarrow & \quad \langle \text{Weakening, } P \Rightarrow P \vee Q \rangle \\ & P \vee Q \end{array}$$

But in the following example, the hint is not precise. This is because the soundness of the step depends not only on Weakening, $P \Rightarrow P \vee Q$, but also on Monotonic $\wedge$ (2).

$$\begin{array}{ll} & P \wedge R \\ \Rightarrow & \quad \langle \text{Weakening, } P \Rightarrow P \vee Q \rangle \\ & (P \vee Q) \wedge R \end{array}$$

We seek a uniform way of substantiating steps like the above one. Rather than rely directly on all the individual monotonicity properties (1)–(9), it is easier to rely on Metatheorem Monotonicity, which can be used to substantiate almost all such weakening/strengthening steps.

We suggest the use of "Monotonicity:" and "Antimonotonicity:" to show reliance on this metatheorem, as shown below. The word "Monotonicity" suggests that the parity of the position of the subexpression involved in the replacement is even, so that the step is a weakening one. Similarly, the word "Antimonotonicity" suggests that the parity of the position involved in the replacement is odd, so that the step is a strengthening one. In the examples given below, to the right we have shown how the formulas can be rewritten in terms of variable z, so that the use of the metatheorem is more easily seen.

$$\begin{array}{lll} & (\forall x \mid : P \wedge R) & (\forall x \mid : z \wedge R)[z := P] \\ \Rightarrow & \quad \langle \text{Monotonicity: Weakening } P \Rightarrow P \vee Q \rangle & \\ & (\forall x \mid : (P \vee Q) \wedge R) & (\forall x \mid : z \wedge R)[z := P \vee Q] \end{array}$$

$$\begin{array}{lll} & \neg(\forall x \mid \neg P \wedge R : S) & \neg(\forall x \mid \neg z \wedge R : S)[z := P] \\ \Leftarrow & \langle \text{Antimonotonicity: Weakening } P \Rightarrow P \vee Q \rangle & \\ & \neg(\forall x \mid \neg(P \vee Q) \wedge R : S) & \neg(\forall x \mid \neg z \wedge R : S)[z := P \vee Q] \end{array}$$

Discussion

Monotonicity properties (1)–(9), as well as metatheorem Monotonicity, are well-known. They can be found, in one guise or another, in several texts on logic. But the two major books that deal with the calculational approach do a bad job of explaining how monotonicity/antimonotonicity is to be used. On page 61 of [1], Dijkstra and Scholten discuss the monotonic properties of negation and implication. But they don't state the general theorem (11) and they don't give a good method of explaining when it is being used. On page 93 of [1], a hint explicitly states the use of monotonicity of $\wedge$ and $\exists$ in a weakening step, but on pages 73 and 77, monotonicity of $\forall$-body is used without mention. I think the problem is that the authors just didn't realize that monotonicity would be a problem for many people.

Gries and Schneider [4] also do not treat montonicity well, and this has resulted in confusion among students about monotonicity and its use. The next edition of [4] is expected to use the approach of this note in order to eliminate the confusion. In Section 4.1 of [4], which introduces weakening/strengthening steps in calculations, parity, metatheorem monotonicity (restricted to propositional calculus), and the new kind of hint will be introduced and explained.

Introduction for EA $\Rightarrow$ AE

Carroll Morgan [5] derives a nice proof of

(12) $(\exists x \mid : (\forall y \mid : P.x.y)) \Rightarrow (\forall y \mid : (\exists x \mid : P.x.y))$

and Wim Feijen [3] presents convincing heuristics for the development of the proof. Here is the proof.

$$\begin{array}{ll} & (\exists x \mid : (\forall y \mid : P.x.y)) \\ \Rightarrow & \langle R \Rightarrow (\forall y \mid : R)\text{, provided } y \text{ does not occur free in } R \text{ —a previous theorem.} \\ & \quad \text{Introduce the necessary universal quantification over } y \rangle \\ & (\forall y \mid : (\exists x \mid : (\forall y \mid : P.x.y))) \\ \Rightarrow & \langle \text{Monotonicity: Instantiation —Eliminate universal quantification} \rangle \\ & (\forall y \mid : (\exists x \mid : P.x.y)) \end{array}$$

Dijkstra [2] discusses the same theorem but with non-*true* ranges:

(13) $(\exists x \mid R.x : (\forall y \mid S.y : P.x.y)) \Rightarrow (\forall y \mid S.y : (\exists x \mid R.x : P.x.y))$.

But now the proof gets messier, because of the non- *true* ranges. The purpose of Dijkstra's note [2] was to present a way of dealing with such non- *true* ranges. Dijsktra introduces the notion of "Range diffusion" for punctual functions and predicates and proves (13) using it.

Here, we present an alternative proof of (13), which rests on Theorem Witness (9.30) of [4] (square brackets denote universal quantification over the state space):

(14) **Theorem Witness.** Suppose $\hat{x}$ does not occur free in R, P, and Q. Then

$$[(\exists x \mid R.x : P.x) \Rightarrow Q.x] \equiv [R.\hat{x} \wedge P.\hat{x} \Rightarrow Q.x]$$

Further, if x does not occur free in Q, then

$$[(\exists x \mid R.x : P.x) \Rightarrow Q] \equiv [R.x \wedge P.x \Rightarrow Q]$$

(The symbol $\hat{x}$ is called a *witness* for the existential quantification.)

The use of Theorem Witness strips away an existential quantification, and this may make dealing with ranges easier and simplify a proof in other ways as well. We illustrate this with a proof of (13).

By Theorem Witness, (13) is a theorem precisely when the following is:

$$R.x \wedge (\forall y \mid S.y : P.x.y) \Rightarrow (\forall y \mid S.y : (\exists x \mid R.x : P.x.y)) \quad .$$

So we prove this formula.

$R.x \wedge (\forall y \mid S.y : P.x.y)$
$\Rightarrow$ ⟨Predicate calculus —since the consequent is a universal quantification over y⟩
$(\forall y \mid S.y : R.x \wedge P.x.y)$
$\Rightarrow$ ⟨Monotonicity: $\exists$-Introduction⟩
$(\forall y \mid S.y : (\exists x \mid : R.x \wedge P.x.y))$
$=$ ⟨Trading⟩
$(\forall y \mid S.y : (\exists x \mid R.x : P.x.y))$

References

[1] Edsger W. Dijsktra and Carel S. Scholten. *Predicate Calculus and Program Semantics.* Springer Verlag, New York, 1990.

[2] Edsger W. Dijsktra. Triggered by Wim Feijen's treatment of "$\exists\forall \Rightarrow \forall\exists$". EWD1201, 27 February 1995.

[3] Wim H.J. Feijen. $\exists\forall \Rightarrow \forall\exists$. WHJ189, September 1994.

[4] David Gries and Fred B. Schneider. *A Logical Approach to Discrete Math.* Springer Verlag, New York 1993.

[5] Carroll Morgan. Email communications in September 1995.