NUMERICAL ANALYSIS Q EXAM

January 17, 1996

The various parts of these problems are not strongly coupled.
If in doubt, move on to the next part to maximize your partial credit.

1. Your friend from the Astronomy department has been taking photographs of a distant galaxy and wants to approximate its shape by an ellipse. Specifically, suppose you have two column vectors x and y of dimension m describing coordinates of points (x_i, y_i). These points lie approximately on an ellipse

$$c_1 x^2 + c_2 xy + c_3 y^2 = 1. \qquad (*)$$

The problem is, what are good values of the coefficients c_i?

Finding coefficients c_i that minimize the least-squares error between the data and an ellipse is a nonlinear problem. Instead of doing that, let's just solve a linear least-squares problem to minimize the square of the residual in the equation (*). (The solution might correspond to a parabola or a hyperbola, if the data are particularly poor, but don't worry about this.)

(a) Write down the linear least-squares problem just described in standard matrix and vector notation for such problems.

(b) One method of solving this problem involves QR factorization. Briefly explain this method, stating clearly what a QR factorization is and how it is used for the least-squares problem. You do not need to explain how a QR factorization is computed.

(c) Another method of solving the problem involves Cholesky factorization. Again, briefly explain this method, stating clearly what a Cholesky factorization is and how it is used for the least-squares problem. You do not need to explain how a Cholesky factorization is computed.

(d) Comment briefly on the relative merits of QR and Cholesky methods for linear least-squares problems.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a given 3-times differentiable function that we would like to minimize starting from a given $\mathbf{x}_0 \in \mathbb{R}^n$. A class of methods proposed in the literature tries to solve this by following the trajectory given by the initial value problem (IVP)

$$\frac{d\mathbf{x}}{dt} = -\nabla f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

Assume that numerical methods in the following questions never reach a point $\mathbf{x}$ where $\nabla f(\mathbf{x}) = \mathbf{0}$ exactly.

(a) Suppose that f is *strictly convex*, meaning that its Hessian at every point is positive definite. Show that in this case, the IVP is *stable*. (Use the definition of stability based on eigenvalues.)

(b) Show that if the Euler method is applied to this IVP, the step it produces always corresponds to a descent direction for f.

(c) Consider the special case where f is a one-variable quadratic $f(x) = ax^2 + bx + c$ with $a > 0$, so the minimizer is $x^* = -b/(2a)$. Show that the *backward* Euler method with a fixed time step converges linearly to the minimizer for any positive time step.