Q-Exam 1995 (Numerical Analysis)

1.(a) (6pts) Outline an efficient method for solving a linear system of the form

$$\begin{bmatrix} A_{11} & A_{12} & R \\ A_{21} & A_{22} & 0 \\ R^T & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \\ d \end{bmatrix}.$$

Here, $R \in \mathbf{R}^{p\times p}$ is nonsingular and upper triangular and $A_{22} \in \mathbf{R}^{m\times m}$ is symmetric positive definite.

(b) (10pts) Outline a method for solving the linear system

$$\begin{bmatrix} A & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

where $A \in \mathbf{R}^{n\times n}$ is positive definite, $C \in \mathbf{R}^{n\times p}$ has rank p, $b \in \mathbf{R}^n$, and $d \in \mathbf{R}^p$. Assume that the QR factorization of C has been computed.

(c) (4pts) Using norms, explain what it means for a matrix to be *nearly rank deficient* with respect to the floating point precision. Explain why the matrix

$$\begin{bmatrix} A & C \\ C^T & 0 \end{bmatrix}$$

in part **(b)** is nearly singular if C is nearly rank deficient.

2. Assume that

$$F(x) = \begin{bmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{bmatrix} \qquad F_i : \mathbf{R}^n \to \mathbf{R}$$

is continuously differentiable. Assume the availability of a function **F** that can be used to evaluate F.

(a) (6pts) Briefly describe the computations associated with a step of the finite difference Newton method when it is applied to finding a zero for F.

(b) (6pts) Why is the machine precision an issue when computing the finite differences?

(c) (8pts) Assume that for all x,

$$\frac{\partial F_i(x)}{\partial x_j} = 0$$

whenever $|i - j| > 1$. Explain why in this special case, only four F-evaluations are required for a finite difference Newton step regardless of n. Hint: A well-chosen direction in a divided difference can yield a lot of partial derivative information.