CS 421: Numerical Analysis
Fall 2000
Final Exam

Handed out: Friday., Dec. 8.

This exam has 12 questions and 120 points total on 3 pages (including this page). You have 120 minutes to complete all the questions. Write your answers in the booklet. This exam is closed-book and closed-note, but you may consult a prepared sheet of notes (one page, $8\frac{1}{2}'' \times 11''$ written on both sides).

If you are a Computer Science PhD student who is taking this exam as the NA Q-exam, write your code number on the booklet instead of your name.

Everyone else: please write your name on the booklet.

The following questions are short answer—a single phrase or formula suffices.

1. [**5 points**] How many flops, accurate to the leading term, are required for back substitution to solve the upper triangular system $U\mathbf{x} = \mathbf{b}$, $U \in \mathbf{R}^{n\times n}$ and $\mathbf{b} \in \mathbf{R}^n$?

2. [**5 points**] Write down the formula for the 1-norm of a vector $\mathbf{x} \in \mathbf{R}^n$.

3. [**5 points**] What is the relationship between $\text{cond}(A)$ and $\text{cond}(2A)$, where "cond" denotes the 2-norm condition number and A is an $n \times n$ nonsingular matrix?

4. [**5 points**] If the eigenvalues of $A \in \mathbf{R}^{3\times 3}$ are $6, -2, 0$, then what are the eigenvalues of $(I - A)^{-1}$?

5. [**5 points**] Given $A \in \mathbf{R}^{m\times n}$, $m \geq n$, and $\mathbf{b} \in \mathbf{R}^m$, a standard algorithm for minimizing $\|A\mathbf{x} - \mathbf{b}\|_2$ is QR-factorization followed by back-substitution. This algorithm fails if $\text{rank}(A) < n$. Specifically, what goes wrong?

6. [**5 points**] Let Q be an $n \times n$ orthogonal matrix. What are its singular values?

7. [**5 points**] Name a method that could be used to solve for the next iterate (i.e., for $\mathbf{y}^{(k+1)}$) of the Backward Euler method.

8. [**5 points**] The state of Florida has been in the news a lot during the past month. How many letters are in the word *Florida*? It's OK to count and then recount.

These questions require longer answers.

9. **[20 points]** Suppose $A \in \mathbf{R}^{n \times n}$ is a square nonsingular matrix. Suppose one is given $\mathbf{b} \in \mathbf{R}^n$ and entries $2, \ldots, n$ of a vector $\mathbf{x} \in \mathbf{R}^n$. Consider the problem of choosing $x(1)$ to minimize $\|A\mathbf{x} - \mathbf{b}\|_2$ given all the other data. Computing $x(1)$ is a linear least-squares problem. Write down the normal equations of this linear least-squares problem.

10. **[20 points]** Construct a polynomial function $p(x)$ such that Newton's method for finding the roots $p(x)$ starting at $x^{(0)} = 0$ fails to converge, but rather generates the sequence $x^{(0)} = 0$, $x^{(1)} = 1$, $x^{(2)} = 0$, $x^{(3)} = 1$, etc. [Hint: there is a degree-2 solution.]

11. **[20 points]** Let $G \in \mathbf{R}^{2 \times 2}$ be the Givens rotation $[\cos\theta, \sin\theta; -\sin\theta, \cos\theta]$. Assume that θ is not a multiple of π. Argue that G cannot have any real eigenvalues. How well would you expect the (unshifted) power method for finding eigenvectors of G, initialized with a real vector, to work?

12. **[20 points]** Consider a scalar initial value problem of the form $dy/dt = f(y)$ where $y(0) = y_0$. Suppose $y_0 > 0$, and suppose f is a decreasing, continuously-differentiable function such that $f(0) = 0$. (So, in particular, $f(z) < 0$ for $z > 0$ and $f(z) > 0$ for $z < 0$.) It is a known theorem that the true solution to this IVP is a positive-valued function. If the Euler method is applied to this IVP, is the computed solution guaranteed to be positive-valued? How about the Backward Euler method? Explain your answers.