Part II
Analytic approximation of some exotic options

1 Introduction

In this chapter we apply the method of lines to approximate values of several options of both European and American styles. In particular, we apply this method to options on stock with constant continuous dividend pay-out and to Asian\(^1\) options. The method is analytic in its nature. We find an exact general solution to a recursive system of non-homogeneous ordinary differential equations (ODE’s) that approximates values of these options. However, we need to use a numerical procedure to select certain coefficients from the boundary conditions. As the actual tests show, this method appears to converge to the unknown exact values of the options in a computationally efficient manner.

An option is a financial security that gives to a purchaser the right to buy or, depending on the contract, the right to sell a fixed number of shares of a particular stock for a certain price at some future date or dates. If this right is used on a particular date, then it is common to say that the option is exercised. The amount of money that a purchaser has to pay for the contract

\(^1\)Asian options are sometimes referred to as integral options or options on average.
is called the price of an option. Options can be of European or American styles. The distinction between European and American styles of options can be made according to the rules for the option’s exercise. An exercise of a European option is allowed only at a maturity date specified by the contract. An American option exercise can occur at any date until its expiration that is also referred to as a maturity date.

Under certain assumptions, an option value can be described by a partial differential equation (PDE)\(^2\). If two options are identical except that one is European and the other is American, then their values are described by the same differential equations. The only difference is in the corresponding boundary conditions. To value an option, one can try to solve the corresponding PDE under appropriate boundary conditions. In many cases, however, an analytic solution can not be found unless some approximations are made. Numerical methods such as finite differences and Monte Carlo simulation can be applied in situations where neither analytic solutions nor reasonable analytic approximations are available.

No exact analytic solution is known for the PDE’s for options on stocks with constant continuous dividend pay-out and for Asian options based on the arithmetic mean. The papers related to the subject of options on dividend paying stocks consider discrete dividends [3], [9], [13], [14] or continuous proportional dividends [4]. Merton [12] considered the valuation of options on stocks with constant continuous dividend pay-out. He gave a sufficient condition for no early exercise of American options and also valued perpetual options. However, he was unable to value or approximate finite-lived options. The case of Asian options based on the geometric mean is completely solved in [6] and [10]. The perpetual case of arithmetic mean options is solved in [11]. In some papers analytic approximations are developed for finite-lived Asian options based on the arithmetic mean but these results do not provide any indication of their precision. Eydeland and Geman [7] suggest a numerical procedure to invert the Laplace Transformation of arithmetic mean options developed in [8]. Even though the options on average asset values could be approximated by the method of Monte Carlo [10], accurate analytic estimation of these options is still an open question.

To value options that do not have an exact analytic solution, one can try the method of lines that was first introduced to the finance literature by P.

Carr and D. Faguet [5]. The main idea of this method is to approximate a partial derivative with respect to time in the corresponding PDE by a finite difference and then to solve the resulting sequence of non-homogeneous ODE’s. The approximation error of this method is determined by the length of time increments used for finite differences. Richardson extrapolation can be applied to accelerate convergence of the approximation errors to zero.

In [5], P. Carr and D. Faguet obtained a closed form solution to the sequence of ODE’s approximating American options on stocks that do not pay dividends. They also have a semi-analytic approximation for American options on stocks paying continuous proportional dividends. We will present a general analytic solution for a sequence of ODE’s approximating options on stock with constant continuous dividend pay-out and Asian options based on the arithmetic average. These two types of options are approximated by similar systems of differential equations3. In section 5 we find their general solution. This solution is a generalization of the confluent hyper-geometric functions. To value an option in a particular case, we suggest an algorithm that utilizes the general solution and uses numerical methods to incorporate the effect of the boundary conditions in the solution. Although the obtained solution does not have a closed form, it has the nice property that in practice it can be applied to both European and American styles of the options. Our general result could also be used to value some interest rate sensitive options.

2 Preliminaries

This section explains our terminology and reviews some basic properties of a recursive system of non-homogeneous ODE’s of second order. Such a system arises in our approximation scheme. Suppose that \( \mathcal{L}(\cdot) \) is a second order linear differential operator defined on functions in \( \mathcal{C}^2 \). For example we can take a second order linear operator \( \mathcal{L}(\cdot) \) defined by the equation

\[
\mathcal{L}(Y) := S^2 Y_{ss} + (2S - 3) Y_s - 4Y
\]

where \( Y \) is a twice continuously differentiable function of \( S \) and \( Y_s, Y_{ss} \) are correspondingly the first and the second derivatives of \( Y \). The operator \( \mathcal{L}(\cdot) \) is of the second order because it contains the second derivative of \( Y \).

\(^3\)Compare the systems of equations in Lemmas 3 and 5.
The operator $\mathcal{L}(\cdot)$ can be used to define a recursive system of non-homogeneous second order ODE’s

$$\mathcal{L}(Y^{(n)}) = \lambda \cdot Y^{(n-1)}, \quad n \geq 1$$

(1)

where $\{Y^{(n)}\}_{n=0}^{\infty}$ is a sequence of functions in $C^2$ and $\lambda$ is a known constant. Note that the non-homogeneous part of the $n^{th}$ equation (1) contains the function $Y^{(n-1)}$, that is, the solution of this recursive system from the previous iteration. To complete the description of the system we must specify the initial condition

$$Y^{(0)} = U$$

(2)

where $U$ is a given function of $S$.

The general solution of the system (1), (2) can be obtained as follows. Suppose that functions $A_0$ and $B_0$ are any two linearly independent solutions of the corresponding homogeneous equation

$$\mathcal{L}(Y) = 0.$$ 

Also assume that we know one particular sequence of functions $\{U_n\}_{n=0}^{\infty}$ that solves (1) under initial condition (2). That is, $U_0 = U$ and

$$\mathcal{L}(U_n) = \lambda \cdot U_{n-1}, \quad \forall \ n \geq 1.$$ 

Then the general solution of equation (1) for $n = 1$ is

$$Y^{(1)} = \alpha_1 A_0 + \beta_1 B_0 + U_1$$

where $\alpha_1$ and $\beta_1$ are arbitrary real numbers. To figure out the general solution of equation (1) for $n = 2$, we must solve the non-homogeneous ODE

$$\mathcal{L}(Y^{(2)}) = \lambda \left( \alpha_1 A_0 + \beta_1 B_0 + U_1 \right).$$

Suppose that some functions $A_1$ and $B_1$ satisfy $\mathcal{L}(A_1) = A_0$ and $\mathcal{L}(B_1) = B_0$. Then the general solution of (1) for $n = 2$ is

$$Y^{(2)} = (\alpha_2 A_0 + \beta_2 B_0) + \lambda (\alpha_1 A_1 + \beta_1 B_1) + U_2$$

where $\alpha_2$ and $\beta_2$ are arbitrary constants. The following lemma extends this to an arbitrary $n$. 

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Lemma 1 To specify the general solution of the recursive system of ODE’s (1) under initial condition (2) it is sufficient to find some sequences of functions \( \{A_n\}, \{B_n\}, \) and \( \{U_n\} \) for \( n \in \{0, 1, 2, \ldots \} \) such that

(a) \( A_0 \) and \( B_0 \) are independent solutions of \( \mathcal{L}(Y) = 0 \)

(b) \( U_0 = U \)

(c) \( \mathcal{L}(A_n) = A_{n-1}, \mathcal{L}(B_n) = B_{n-1}, \) and \( \mathcal{L}(U_n) = \lambda U_{n-1} \) for any \( n \geq 1 \).

If such functions are available then the general solution of (1), (2) is

\[
Y^{(n)} = \sum_{i=1}^{n} \alpha_i \lambda^{n-i} A_{n-i} + \sum_{i=1}^{n} \beta_i \lambda^{n-i} B_{n-i} + U_n \quad \forall n \geq 0
\]

where coefficients \( \alpha_i \) and \( \beta_i \) are arbitrary real numbers.

**Proof:** The induction argument gives

\[
\mathcal{L}(Y^{(n)}) = \sum_{i=1}^{n} \alpha_i \lambda^{n-i} \mathcal{L}(A_{n-i}) + \sum_{i=1}^{n} \beta_i \lambda^{n-i} \mathcal{L}(B_{n-i}) + \mathcal{L}(U_n) =
\]

\[
= \sum_{i=1}^{n-1} \alpha_i \lambda^{n-i} A_{n-1-i} + \sum_{i=1}^{n-1} \beta_i \lambda^{n-i} B_{n-1-i} + \lambda U_{n-1} =
\]

\[
= \lambda Y^{(n-1)}
\]

and the proof is done. \( \blacksquare \)

Recursive systems of non-homogeneous second order ODE’s similar to (1), (2) will be used in the following sections to approximate PDE’s arising in the option valuation problems. Lemma 1 determines the structure of the general solutions of these approximating systems.

3 Options on stock with constant dividends

In this section we consider options on a stock that pays a constant continuous dividend. First, we derive the PDE that corresponds to options of this type. Then we approximate this PDE by a recursive sequence of non-homogeneous ODE’s.
Suppose that \( t \) represents continuous time and \( S \) stands for a current underlying stock price. If \( d \) denotes a dividend pay-out over a unit of time then our model for the stock price evolution is given by the following stochastic differential equation

\[
\begin{align*}
    dS & = S \mu dt + S \sigma dW - d dt & \text{if } S > 0 \\
    dS & = 0 & \text{if } S = 0,
\end{align*}
\]

where \( W \) is a Brownian motion, \( \mu \) is an average growth rate, and \( \sigma \) is a parameter that represents volatility of the underlying stock. Note that this model allows for bankruptcy in contrast to most other models of price processes for dividend paying stocks\(^4\).

To be specific, we consider a standard European call option on a stock that follows (3). The main property of the European call option is that it gives to its holder the right to buy one share of stock at a fixed price \( K \) when the option reaches its maturity date \( T \). The price \( K \) and the maturity \( T \) are specified at the contract initiation. \( K \) is usually referred to as the \textit{strike} price. Suppose that function \( C(\tau, S) \) denotes the value of the call when there are \( \tau = T - t \) units of time left until the option’s maturity and if the current stock price is \( S \). The value of the European call option at its maturity is\(^5\)

\[
C(0, S) = (S - K)^+ = \max(0, S - K). \tag{4}
\]

There are other conditions which the value of the European call has to satisfy:

\[
\begin{align*}
    C(\tau, 0) & = 0, \tag{5} \\
    C(\tau, S) & = O(S) \quad \text{as } S \to +\infty. \tag{6}
\end{align*}
\]

Equations (4), (5), and (6) are the boundary conditions for \( C(\tau, S) \).

We now derive a PDE for the function \( C(\tau, S) \).\(^6\) From Itô’s rule and from (3), we conclude that for \( S > 0 \) and for \( 0 \leq \tau = (T - t) \leq T \) the call value

\(^4\)(3) is also a reasonable description for the value of the assets of a firm which has debt outstanding. If the debt accrues interest continuously and the assets are liquidated continuously to finance the interest pay-out of \( d \), then (3) describes the asset value until the debt is fully paid off.

\(^5\)In fact, condition (4) holds for American calls too.

\(^6\)We follow Merton’s paper [12].
\[ C(T - t, S) \text{ satisfies} \]
\[
dC = -C_r dt + C_s dS + \frac{1}{2} C_{ss} S^2 \sigma^2 dt =
\]
\[= -C_r dt + C_s S \mu dt + C_s S \sigma dW - C_s d dt + \frac{1}{2} S^2 \sigma^2 C_{ss} dt. \quad (7)\]

At time \( t \) we can form a portfolio that consists of one call and \(-C_s(T - t, S)\) shares of stock. The value of this portfolio is
\[ V = C - C_s S. \]

Since the stock pays constant continuous dividends then at time \( t \)
\[ dV = dC - C_s (dS + d dt) = dC - C_s (S \mu dt + S \sigma dW). \]

As follows from (7)
\[ dV = \left( -C_r - C_s d + \frac{1}{2} S^2 \sigma^2 C_{ss} \right) dt. \]

Therefore, at time \( t \) the portfolio \( V \) is riskless and the assumption of \textit{no arbitrage} would imply that it should grow at the same rate as a riskless bank account would grow. That is,
\[ dV = r V dt = r \left( C - C_s S \right) dt \]

where \( r \) is an interest rate that we assume to be a known constant parameter. The last two equations imply that \( C(\tau, S) \) satisfies the PDE obtained by Merton in [12]
\[ \frac{1}{2} S^2 \sigma^2 C_{ss} + (rS - d) C_s - rC = C_\tau. \quad (8)\]

Note that the same PDE works for American style calls\(^7\) and for put options\(^8\) as long as the underlying stock follows model (3).

We will use PDE (8) as a starting point of our search for the value of the corresponding options. Choose an arbitrary positive integer \( N \) and take

\(^7\) The domain of this PDE for American style options is the continuation region.

\(^8\) A standard put is a right to sell one share of some stock at a fixed strike \( K \). The value of a put upon its exercise is \((K - S)^+\).
\[ \Delta \tau = \frac{T}{N}. \]  
We will concentrate on the values of \( C(\tau, S) \) at times \( \tau = n\Delta \tau \)

where \( n \) is a nonnegative integer which is less than or equal to \( N \). Let \( C^{(n)}(S) \)

denote our approximation of \( C(n\Delta \tau, S) \):

\[ C^{(n)}(S) \approx C(n\Delta \tau, S), \quad \forall n \in \{0, \ldots, N\}. \]

The method of lines suggests the approximation

\[ C_{\tau}(n\Delta \tau, S) \approx \frac{C^{(n)}(S) - C^{(n-1)}(S)}{\Delta \tau}. \]

Then (8) yields the recursive system of non-homogeneous ODE’s

\[ S^2C^{(n)}_{ss} + (\gamma S - \delta)C^{(n)}_s - (\gamma + \eta)C^{(n)} = -\eta C^{(n-1)} \]  \hspace{1cm} (9)

where

\[ \gamma = \frac{2r}{\sigma^2}, \quad \delta = \frac{2d}{\sigma^2}, \quad \eta = \frac{2}{\sigma^2 \Delta \tau}, \quad n \in \{1, \ldots, N\}. \]

From (4) we know that the initial condition is

\[ C^{(0)}(S) = (S - K)^+ . \]  \hspace{1cm} (10)

Since the total number of time steps \( N \) could be arbitrarily large then

we should solve (9) for all integers \( n \geq 1 \). The structure of the general

solution of the system (9), (10) is given by Lemma 1. First, we will find

one particular sequence of functions \( \{U_n\} \) that satisfies equation (9) and the

initial condition

\[ U_0(S) = \begin{cases} 
    S - K & \text{if} \quad S \geq K \\
    0 & \text{if} \quad 0 < S < K.
\end{cases} \]

The general solution of the system (9), (10) could be found separately on

the intervals \( 0 < S < K \) and \( S \geq K \). On the interval \( S \in (0, K) \) we can

obviously take \( U_n = 0 \) for all \( n \). The next lemma provides us with a sequence

of functions that works for \( S \in [K, +\infty) \).

**Lemma 2** The sequence of linear functions \( C^{(n)}_U(S) = S + u_n \) where

\[ u_0 = -K, \quad u_n = \frac{u_{n-1} - d \Delta \tau}{1 + r \Delta \tau} \quad (n \geq 1) \]

satisfies equation (9) and the initial condition \( C^{(0)}_U(S) = S - K \).
PROOF: Substitute the formula for $C_U^{(n)}$ into (9) and use induction. ■

Lemma 1 implies that we also have to find two sequences of functions \{\(C_A^{(n)}\)\} and \{\(C_B^{(n)}\)\} such that \(C_A^{(0)}\) and \(C_B^{(0)}\) are independent solutions of the homogeneous ODE

\[
S^2 C_A^{(0)} + (\gamma S - \delta) C_A^{(0)} - (\gamma + \eta) C^{(0)} = 0
\]

and such that each of these two sequences satisfies the recursive equation

\[
S^2 C_A^{(n)} + (\gamma S - \delta) C_A^{(n)} - (\gamma + \eta) C_A^{(n-1)} = C_A^{(n-1)}, \quad n \geq 1.
\]

Specific formulas for functions \(C_A^{(n)}\) and \(C_B^{(n)}\) will be obtained in the end of section 5\(^9\). The following lemma provides a substitution that transforms the system of equations (11), (12) to a standard form.

**Lemma 3** Suppose that \(z = \frac{\delta}{S}\) and \(C^{(n)}(\frac{\delta}{S}) = e^{-z} z^p h^{(n)}(z)\) where \(p\) is either \(p_1\) or \(p_2\):

\[
p_{1,2} = \left(\frac{\gamma - 1}{2}\right) \pm \sqrt{\left(\frac{\gamma - 1}{2}\right)^2 + (\gamma + \eta)}.
\]

Then the sequence \(\{C^{(n)}\}\) satisfies equation (12) and the initial condition (11) if and only if the sequence \(\{h^{(n)}\}\) satisfies the recursion

\[
z h_z^{(n)} + (b - z) h_z^{(n)} - a h^{(n)} = \frac{h^{(n-1)}}{z}
\]

and the initial condition

\[
z h_z^{(0)} + (b - z) h_z^{(0)} - a h^{(0)} = 0
\]

where \(a = p + 2 - \gamma\) and \(b = 2p + 2 - \gamma\).

PROOF: It is easy to check that if \(z = \frac{\delta}{S}\) and \(C^{(n)}(S) = e^{-z} z^p h^{(n)}(z)\) then

\[
C_A^{(n)}(S) = -\frac{z^{p+1}}{\delta} e^{-z} \left( z h^{(n)}_z(z) + (p - z) h^{(n)}(z) \right)
\]

\[
C_A^{(n-1)}(S) = -\frac{z^{p+2}}{\delta^2} e^{-z} \left[ z^2 h^{(n)}_z(z) + 2z(p + 1 - z) h^{(n)}_z(z) + (z^2 + (p - 2z)(p + 1)) h^{(n)}(z) \right].
\]

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\(^9\)See equations (49) and (50).
Substituting these formulas into (12) gives
\[ z^2 h_x^{(n)} + z h_z^{(n)}(b - z) + h^{(n)}(-az + p^2 + p(1 - \gamma) - (\gamma + \eta)) = h^{(n-1)}. \]
This equation completes the proof of Lemma 3 since \( p_{1,2} \) are the roots of the quadratic equation \( p^2 + p(1 - \gamma) - (\gamma + \eta) = 0. \)

\[ \square \]

4 Asian options

Consider now an Asian option based on the arithmetic mean. To be specific we consider a European style put struck at the average. This option is a financial security which gives to an owner a right to sell one share of some stock upon maturity \( T \) for the average price that this stock exhibits from the contract initiation until its expiration at \( T \). For simplicity, we will assume that the stock does not pay any dividends\(^{10}\) and that its price satisfies the stochastic model
\[ dS = S \mu dt + S \sigma dW \quad \text{for } S > 0. \]

Suppose that \( I(t) \) denotes the following integral:
\[ I(t) = \int_0^t S(t')dt'. \]
One can show\(^{11}\) that the value of the put on the average is a function of the time to maturity \( \tau = T - t \), the stock price \( S \), and the integral \( I \). We will denote this function by \( A(\tau, S, I) \). The value of this option at maturity \( \tau = 0 \) is
\[ A(0, S, I) = \left( \frac{I}{T} - S \right)^+. \]

Ito’s rule and equation (15) imply that for \( 0 \leq \tau = T - t \leq T \)
\[
\begin{align*}
daA &= -A_{\tau} dt + A_t dI + A_S dS + \frac{1}{2} S^2 \sigma^2 A_{ss} dt \\
&= -A_{\tau} dt + A_t S dt + A_S S \mu dt + A_S S \sigma dW + \frac{1}{2} S^2 \sigma^2 A_{ss} dt.
\end{align*}
\]
\(^{10}\)Continuous proportional dividends are easily handled.\(^{11}\)See [15] for reference.
At time $t$ we form a portfolio that consists of one option and $-A_s(T-t, S, I)$ shares of stock. The value $Z$ of this portfolio is

$$Z = A - A_s S.$$  

We see that at time $t$

$$dZ = dA - A_s dS = \left(-A_r + A_1 S + \frac{1}{2} S^2 \sigma^2 A_{ss}\right) dt.$$

Applying the same *no arbitrage* condition as before, we obtain

$$dZ = rZ dt = r (A - A_s S) dt.$$

The last two equations yield the following PDE for $A(\tau, S, I)$:

$$SA_1 + r S A_s + \frac{1}{2} S^2 \sigma^2 A_{ss} - r A = A_r.$$

It is convenient to make a substitution

$$A(\tau, S, I) = S \cdot V(\tau, \psi), \quad (17)$$

where $\psi = \frac{S}{S}$. The function $V(\tau, \psi)$ satisfies the PDE

$$\frac{1}{2} \psi^2 \sigma^2 V_{\psi \psi} + (1 - r \psi) V_\psi = V_{\tau}. \quad (18)$$

Condition (16) implies that

$$V(0, \psi) = \left(\frac{\psi}{T} - 1\right)^+. \quad (19)$$

Note that equation (18) will work for American style puts struck at the average\textsuperscript{12} and for European and American style calls struck at the average\textsuperscript{13}. An equation similar to (18) is considered by Kramkov and Mordecki in [11]. This paper solves the perpetual case of the integral options where the time derivative is irrelevant. Kramkov and Mordecki obtained the exact analytic solution of (18) in the case where $V_\tau = 0$.

\textsuperscript{12}Domain for the PDE is the continuation region.

\textsuperscript{13}A European call struck at the average pays $(S - \frac{L}{T})^+$ at its exercise at time $\tau = 0$.  

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We will approximate (18) using the method of lines. Take an arbitrary positive integer \( N \) and \( \Delta \tau = \frac{T}{N} \). Let \( V^{(n)}(\psi) \) denote our approximation of \( V(n\Delta \tau, \psi) \):
\[
V^{(n)}(\psi) \approx V(n\Delta \tau, \psi), \quad \forall n \in \{0, \ldots, N\}.
\]
Assume that
\[
\frac{V^{(n)}(\psi) - V^{(n-1)}(\psi)}{\Delta \tau} \approx V'(n\Delta \tau, \psi).
\]
Then (18) yields the following approximation
\[
\psi^2 V'^{(n)}_{\psi\psi} + (\delta - \gamma \psi) V^{(n)}_{\psi} - \eta V^{(n)} = -\eta V^{(n-1)}
\]
where
\[
\gamma = \frac{2r}{\sigma^2}, \quad \delta = \frac{2}{\sigma^2}, \quad \eta = \frac{2}{\sigma^2 \Delta \tau}, \quad n \in \{1, \ldots, N\}.
\]
From (19) we derive the initial condition
\[
V^{(0)}(\psi) = \left( \frac{\psi}{T} - 1 \right)^+.
\]
We want to find a general solution of the recursive system of ODE’s (20), (21) for all integers \( n \geq 1 \). As follows from Lemma 1, we need to find one particular sequence of functions \( \{U_n\} \) that satisfies (20) and the initial condition
\[
U_0(\psi) = \begin{cases} 
\frac{\psi}{T} - 1 & \text{if } \psi > T \\
0 & \text{if } 0 \leq \psi \leq T.
\end{cases}
\]
We will find a general solution of the system (20), (21) separately on the intervals \( \psi \in [0, T] \) and \( \psi \in (T, +\infty) \). On the first interval we obviously can take \( U_n(\psi) = 0 \) for all \( n \). A sequence of functions that works for the interval \( \psi \in (T, +\infty) \) is given in the next lemma.

**Lemma 4** The sequence of linear functions \( V^{(n)}_{U}(\psi) = p_n \psi + q_n \) where
\[
p_0 = \frac{1}{T}, \quad p_n = \frac{p_{n-1}}{1+r\Delta \tau}, \quad q_0 = -1, \quad q_n = q_{n-1} + \frac{p_{n-1}}{1+r\Delta \tau} \quad (n \geq 1)
\]
satisfies equation (20) and the initial condition \( V^{(0)}_{U}(\psi) = \frac{\psi}{T} - 1 \).
Proof: Substitute the formula for $V_U^{(n)}$ into (20) and use induction. 

Lemma 1 also requires some sequences of functions $\{V_A^{(n)}\}$ and $\{V_B^{(n)}\}$ such that $V_A^{(0)}$ and $V_B^{(0)}$ are independent solutions of the homogeneous ODE

$$
\psi^2 V_{\psi \psi} + (\delta - \gamma \psi) V_{\psi} - \eta V = 0
$$

and such that each of these two sequences satisfies equation

$$
\psi^2 V^{(n)}_{\psi \psi} + (\delta - \gamma \psi) V^{(n)}_{\psi} - \eta V^{(n)} = V^{(n-1)}_{\psi \psi}, \quad n \geq 1.
$$

Specific formulas for functions $V^{(n)}$ and $F^{(n)}$ are found in the end of section 5\textsuperscript{14}. The next lemma converts the system of equations (22), (23) to a standard form.

Lemma 5 Suppose that $z = \frac{\xi}{\psi}$ and $V^{(n)}(\frac{\xi}{\psi}) = z^p h^{(n)}(z)$ where $p$ is either $p_1$ or $p_2$:

$$
p_{1,2} = -\left(\frac{\gamma + 1}{2}\right) \pm \sqrt{\left(\frac{\gamma + 1}{2}\right)^2 + \eta}.
$$

Then the sequence $\{V^{(n)}\}$ satisfies equation (23) and the initial condition (22) if and only if the sequence $\{h^{(n)}\}$ satisfies the recursion

$$
z h^{(n)}_{zz} + (b - z) h^{(n)}_z - a h^{(n)} = \frac{h^{(n-1)}}{z}
$$

and the initial condition

$$
z h^{(o)}_{zz} + (b - z) h^{(o)}_z - a h^{(o)} = 0
$$

where $a = p$ and $b = 2p + 2 + \gamma$.

Proof: It is easy to check that if $z = \frac{\xi}{\psi}$ and $V^{(n)}(\psi) = z^p h^{(n)}(z)$ then

$$
V^{(n)}_{\psi}(\psi) = -\frac{z^{p+1}}{\delta} \left( z h^{(n)}_{z}(z) + ph^{(n)}(z) \right)
$$

$$
V^{(n)}_{\psi \psi}(\psi) = \frac{z^{p+2}}{\delta^2} \left( z^2 h^{(n)}_{zz}(z) + 2z(p+1) h^{(n)}_{z}(z) + p(p+1) h^{(n)}(z) \right).
$$

\textsuperscript{14}See equations (51) and (52).
Substituting these formulas into (23) gives
\[ z^2 h_{zz}^{(n)} + zh_z^{(n)}(b - z) + h^{(n)}(-az + p^2 + p(1 + \gamma) - \eta) = h^{(n-1)}. \]
This equation completes the proof of Lemma 5 since \( p_{1,2} \) are the roots of the quadratic equation \( p^2 + p(1 + \gamma) - \eta = 0 \).

Note that the system of equations (24), (25) is identically the same as the system of equations (13), (14) in Lemma 3.

5 General solution of approximating ODE’s

In this section we establish two sequences of functions \( \{H^{(n)}\} \) and \( \{G^{(n)}\}, \)
\( n \geq 0, \) that satisfy recursive equation (13) and such that \( H^{(0)} \) and \( G^{(0)} \) are independent solutions of the corresponding homogeneous equation (14). As follows from Lemmas 1, 3, and 5, this would complete the search for the general solutions of the recursive systems of non-homogeneous ODE’s (9), (10), and (20), (21).

The homogeneous equation (14)
\[ zh_{zz} + (b - z)h_z - ah = 0 \]
is the well known Kummer’s equation. In Abramowitz and Stegun [1], we find that its independent solutions are
\[ H^{(0)}(z) = M(a, b, z) \] (26)
\[ G^{(0)}(z) = z^{1-b}M(1 + a - b, 2 - b, z). \] (27)

Note that \( M(a, b, z) \) is a standard notation for Kummer’s function\(^\text{15}\):
\[ M(a, b, z) = \sum_{i=0}^{\infty} \frac{(a)_i}{(b)_i} \frac{z^i}{i!} \]
where \( (a)_i = a(a + 1)(a + 2) \ldots (a + i - 1), \quad (a)_0 = 1. \)

Therefore, the problem is to find \( \{H^{(n)}\} \) and \( \{G^{(n)}\}, \) for \( n \geq 1, \) that satisfy the recursive system of ODE’s (13) with the initial elements \( H^{(0)} \) and

\(^{15}\)Kummer’s function belongs to the class of confluent hyper-geometric functions.
\(G^{(0)}\) as in (26) and (27). It is convenient to introduce notation for the linear operator on the left hand side of (13). Denote
\[ \hat{\mathcal{L}} (\cdot) := z(\cdot)_{zz} + (b-z)(\cdot)_z - a(\cdot). \]
Then (13) can be rewritten as
\[ \hat{\mathcal{L}} (h^{(n)}) = \frac{h^{(n-1)}}{z}, \quad n \geq 1. \tag{28} \]

**Theorem 1** Suppose that functions \(f^{(n)}_k, \quad 0 \leq k \leq n\), satisfy the recursive system of equations
\[
\begin{align*}
    f^{(0)}_0(z) &= h^{(0)}(z) = M(a,b,z) \\
    \hat{\mathcal{L}} \left( f^{(n)}_0 \right) &= \mathcal{F}_h \left( f^{(n)}_1 \right) \\
    f^{(n)}_n &= \frac{f^{(n-1)}_{n-1}}{n(b-1)} \\
    f^{(n)}_k &= \frac{f^{(n-1)}_{k-1}}{k(b-1)} - \frac{(k+1)f^{(n)}_{k+1}}{(b-1)} \\
\end{align*}
\]
where \(\mathcal{F}_h(\cdot) := -2(\cdot)_z + (\cdot)\) is a first order linear operator. Then
\[ H^{(n)} = \sum_{k=0}^{n} (\ln z)^k f^{(n)}_k \tag{33} \]
solves the system of equations (26), (28).

**Proof:** First we want to show that equations (29)-(32) imply
\[ \hat{\mathcal{L}} \left( f^{(n)}_k \right) = \begin{cases}
    (k+1)\mathcal{F}_h \left( f^{(n)}_{k+1} \right) & 0 \leq k \leq n-1 \\
    0 & k = n.
\end{cases} \tag{34} \]

Property (34) is obvious for \(k = 0\) due to (30). Equations (29) and (31) imply that \(f^{(n)}_n\) is equal to Kummer’s function \(M(a,b,z)\) multiplied by some constant. Therefore (34) holds for \(k = n\). One can prove (34) for other values
of \( k \) by induction. Assume that \( f^{(n-1)}_k \) satisfies (34) for all \( 0 \leq k \leq n - 1 \). Then check that for \( f^{(n)}_k \) property (34) extends from \( k = n \) to all smaller values of \( k \).

Equation (33) agrees with (26) if \( n = 0 \). For \( n \geq 1 \) we have

\[
\hat{\mathcal{L}} \left( H^{(n)} \right) = (\ln z)^n \hat{\mathcal{L}} \left( f^{(n)}_n \right)
\]

\[
+ (\ln z)^{n-1} \left\{ \hat{\mathcal{L}} \left( f^{(n)}_{n-1} \right) + n \left[ \frac{b-1}{z} f^{(n)}_n + 2 \left( f^{(n)}_n \right)_z - f^{(n)}_n \right] \right\}
\]

\[
+ \sum_{k=0}^{n-2} (\ln z)^k \left\{ \hat{\mathcal{L}} \left( f^{(n)}_k \right) + (k + 1) \left[ \frac{b-1}{z} f^{(n)}_{k+1} + 2 \left( f^{(n)}_{k+1} \right)_z - f^{(n)}_{k+1} \right] 
\]

\[
+ \frac{(k + 1)(k + 2)}{z} f^{(n)}_{k+2} \right\}.
\]

Equations (34), (31), and (32) conclude the proof of (28).

Similar theorem holds for the sequence \( \{G^{(n)}\} \).

**Theorem 2** Suppose that functions \( f^{(n)}_k \), \( 0 \leq k \leq n \), satisfy the recursive system of equations

\[
f^{(0)}_0(z) = g^{(0)}(z) = z^{1-b}M(1 + a - b, 2 - b, z) \quad \text{(35)}
\]

\[
\hat{\mathcal{L}} \left( f^{(n)}_0 \right) = \mathcal{F}_g \left( f^{(n)}_1 \right) \quad \text{n \geq 1} \quad \text{\quad (36)}
\]

\[
f^{(n)}_n = \frac{f^{(n-1)}_{n-1}}{n(1-b)} \quad \text{n \geq 1} \quad \text{\quad (37)}
\]

\[
f^{(n)}_k = \frac{f^{(n-1)}_{k-1}}{k(1-b)} - \frac{(k + 1)f^{(n)}_{k+1}}{(1-b)} \quad 1 \leq k \leq n - 1. \quad \text{\quad (38)}
\]

where \( \mathcal{F}_g(\cdot) := -2z^{1-b} \left( \frac{d}{dz} \right)_z + (\cdot) \) is a first order linear operator. Then

\[
G^{(n)} = \sum_{k=0}^{n} (\ln z)^k f^{(n)}_k \quad \text{\quad (39)}
\]

solves the system of equations (27), (28).
Proof: Repeat the steps of the proof of Theorem 1. The property
\[
\hat{\mathcal{L}}(f_{k}^{(n)}) = \begin{cases} 
(k + 1)\mathcal{F}_{g}(f_{k+1}^{(n)}) & 0 \leq k \leq n - 1 \\
0 & k = n.
\end{cases}
\]
works as an analogue of property (34).

We define the following functions of the argument \( z \):
\[
Y_{k}(a, b, z) := \sum_{m=0}^{\infty} \frac{(a)_{m} z^{m}}{(b)_{m}} \zeta_{m}^{(k)}(a, b) \quad \forall k \geq 0,
\]
(40)
where the coefficients \( \zeta_{m}^{(k)}(a, b) \) solve the recursive equations
\[
\zeta_{0}^{(k)}(a, b) := 1 \quad m \geq 0
\]
\[
\zeta_{m}^{(k)}(a, b) := \begin{cases} 
\sum_{i=0}^{m-1} \left( \frac{\zeta_{i}^{(k-1)}}{a+i} - \frac{2\zeta_{i+1}^{(k-1)}}{b+i} \right) & m \geq 1, \ k \geq 1 \\
0 & m = 0, \ k \geq 1.
\end{cases}
\]

The functions \( Y_{k}(a, b, z) \) are generalizations of Kummer’s function. \( Y_{k} \) will serve as a basic function for constructing sequences \( \{h^{(n)}\} \) and \( \{g^{(n)}\} \) that solve (28).

Lemma 6 The functions \( Y_{k}(a, b, z) \) have the properties:
\[
\begin{align*}
\begin{cases} 
Y_{0}(a, b, z) = h^{(0)} = M(a, b, z) \\
\hat{\mathcal{L}}(Y_{k}(a, b, z)) = \mathcal{F}_{h}(Y_{k-1}(a, b, z))
\end{cases}
\end{align*}
\]
(41)
\[
\begin{align*}
\begin{cases} 
z^{1-b}Y_{0}(a', b', z) = g^{(0)} = z^{1-b}M(a', b', z) \\
\hat{\mathcal{L}}\left(z^{1-b}Y_{k}(a', b', z)\right) = \mathcal{F}_{g}\left(z^{1-b}Y_{k-1}(a', b', z)\right)
\end{cases}
\end{align*}
\]
(42)
where \( a' = 1 + a - b \) and \( b' = 2 - b \).

Proof: We will prove (41). The upper equation is obvious from the definition of \( Y_{0} \). One can check that for \( k \geq 1 \)
\[
\mathcal{F}_{h}(Y_{k-1}(a, b, z)) = \sum_{m=0}^{\infty} \frac{(a)_{m} z^{m}}{(b)_{m}} \frac{m!}{m!} \left( \zeta_{m}^{(k-1)} - \frac{a}{b+m} \zeta_{m+1}^{(k-1)} \right).
\]
From [2] we know that that the function

\[ f(z) = \lambda \frac{z^{\alpha+1}}{(\alpha + 1)(\alpha + b)} \sum_{n=0}^{\infty} \frac{(1 + \alpha + a)_n z^n}{(\alpha + 2)_n (\alpha + 1 + b)_n} \]

solves the differential equation \( \dot{\mathcal{L}}(f) = \lambda \cdot z^\alpha \). Therefore, (41) will be satisfied if we choose

\[ Y_k(a, b, z) = \sum_{m=d}^{\infty} \frac{(a)_m}{(b)_m m!} \left( \zeta_m^{(k-1)} - 2 \frac{a + m}{b + m} \zeta_m^{(k)} \right) \cdot \frac{z^{m+1}}{(m+1)(m+b)} \sum_{n=0}^{\infty} \frac{(a + m + 1)_n z^n}{(m + 2)_n (b + m + 1)_n} = \ldots = \]

\[ = \sum_{m=1}^{\infty} \frac{(a)_m}{(b)_m m!} \sum_{i=0}^{m-1} \left( \frac{\zeta_i^{(k-1)}}{a+i} - \frac{2\zeta_i^{(k-1)}}{b+i} \right) \]

which agrees with the definition of \( Y_k(a, b, z) \) in (40). The proof of (42) is similar to the proof of (41). \[ \blacksquare \]

The following theorem gives functions \( f_k^{(n)} \) that solve the systems of equations in Theorems 1 and 2.

**Theorem 3** Suppose that \( y_k \) is a function of the argument \( z \) and that \( \lambda \) is a real constant. Assume also that functions \( f_k^{(n)} \), \( 0 \leq k \leq n \), satisfy

\[ f_0^{(n)} = \sum_{k=0}^{n} \frac{R_k^{(n)}}{\lambda^k} y_k \quad n \geq 0 \quad (43) \]

\[ f_k^{(n)} = \frac{1}{k! \lambda^k} \sum_{i=0}^{n-k} \frac{(-1)^i C_i^{(k)}}{\lambda^{2i}} f_0^{(n-k-i)} \quad 1 \leq k \leq n \quad (44) \]

where \( C_i^{(k)} \) are the constants defined by formula (58) in the appendix and \( R_k^{(n)} \) are the coefficients defined by the equations

\[ R_0^{(0)} = 1, \quad R_0^{(n)} - \text{any real numbers for } n \geq 1 \]

\[ R_k^{(n)} = \sum_{i=0}^{n-k} \frac{(-1)^i C_i^{(1)}}{\lambda^{2i}} R_{k-1}^{(n-1-i)} \quad 1 \leq k \leq n. \quad (45) \]
If $y_k(z) = Y_k(a, b, z)$ and $\lambda = b - 1$ then functions $f_k^{(n)}$ solve the system of equations (29)-(32) from Theorem 1.

If $y_k(z) = z^{1-b} Y_k(1 + a - b, 2 - b, z)$ and $\lambda = 1 - b$ then functions $f_k^{(n)}$ solve the system of equations (35)-(38) from Theorem 2.

**Proof:** Suppose that $\mathcal{F} := \mathcal{F}_h$ in case of $y_k(z) = Y_k(a, b, z)$ and that $\mathcal{F} := \mathcal{F}_g$ in case of $y_k(z) = z^{1-b} Y_k(1 + a - b, 2 - b, z)$. We have to show that conditions (43) and (44) imply the system of equations

\[
f_k^{(n)} = \frac{f_k^{(n-1)}}{n \lambda} \quad n \geq 1 \tag{46}
\]

\[
f_k^{(n)} = \frac{f_{k-1}^{(n-1)}}{k \lambda} - \frac{(k + 1)f_k^{(n+1)}}{\lambda} \quad 1 \leq k \leq n - 1 \tag{47}
\]

\[
\hat{L} \left( f_0^{(n)} \right) = \mathcal{F} \left( f_1^{(n)} \right) \quad n \geq 1. \tag{48}
\]

Relations (46) and (47) are proved in the appendix. The proof of (48) is based on Lemma 6 which implies the properties

\[
\hat{L} \left( y_0 \right) = 0
\]

\[
\hat{L} \left( y_k \right) = \mathcal{F} \left( y_{k-1} \right) \quad k \geq 1.
\]

Equation (48) follows from these properties by means of induction. \[\square\]

Equations (33), (43), and (44) explicitly define the sequence of functions $\{H^{(n)}\}$ that satisfies (26), (28). Equations (39), (43), (44) specify the sequence of functions $\{G^{(n)}\}$ satisfying (27), (28). The next lemma establishes a simple relationship between the functions $H_n$ and $G_n$.

**Lemma 7** Suppose that $H_n$ and $G_n$ are the functions determined by formulas (33), (39), and by the system of equations from theorem 3. Then

\[
G_n(a, b, z) = z^{1-b} H_n(1 + a - b, 2 - b, z), \quad \forall n \geq 0.
\]

**Proof:** The theorem is trivial for $n = 0$. Consider $n \geq 1$. Suppose that function $f_k^{(n)}(a, b, z)$ satisfies (43), (44) for $y_k = Y_k(a, b, z)$ and $\lambda = b - 1$. Assume also that function $f_k^{(n)}(a, b, z)$ satisfies (43), (44) for $y_k = z^{1-b} Y_k(1 + a - b, 2 - b, z)$ and $\lambda = 1 - b$. Then by (48) we have

\[
\hat{L} \left( f_k^{(n)} \right) = \mathcal{F} \left( f_{k+1}^{(n)} \right)
\]

\[
\frac{f_k^{(n)}}{k \lambda} = \frac{f_{k+1}^{(n+1)}}{\lambda} \quad 1 \leq k \leq n - 1.
\]

By (50) and (51) we have

\[
f_k^{(n)} = \frac{f_{k+1}^{(n+1)}}{k \lambda} \quad 1 \leq k \leq n - 1.
\]

Hence, by (52) and (53) we get

\[
f_k^{(n)} = \frac{f_{k+1}^{(n+1)}}{k \lambda} \quad 1 \leq k \leq n - 1.
\]

\[
G_n(a, b, z) = z^{1-b} H_n(1 + a - b, 2 - b, z), \quad \forall n \geq 0.
\]
\( a - b, 2 - b, z \) and \( \lambda = 1 - b \). As follows from (33) and (39), it suffices to check that
\[
\overline{f}_k^{(n)}(a, b, z) = z^{1-b} f_k^{(n)}(1 + a - b, 2 - b, z).
\]
This relationship easily follows from equations (43), (44), and (45). ■

Finally, Lemmas 3 and 7 imply that the functions
\[
\begin{align*}
C_A^{(n)}(S) &= e^{-z} z^p H^{(n)}(a, b, z) \\
C_B^{(n)}(S) &= e^{-z} z^{p+1-b} H^{(n)}(1 + a - b, 2 - b, z),
\end{align*}
\]
where \( z = \frac{\delta}{S} \), solve equations (11) and (12) from section 3. Using notations in Lemma 3 we equivalently rewrite
\[
\begin{align*}
C_A^{(n)}(S) &= e^{-z} z^{p_1} H^{(n)}(p_1 + 2 - \gamma, 2p_1 + 2 - \gamma, z) \quad (49) \\
C_B^{(n)}(S) &= e^{-z} z^{p_2} H^{(n)}(p_2 + 2 - \gamma, 2p_2 + 2 - \gamma, z). \quad (50)
\end{align*}
\]
Similarly, Lemmas 5 and 7 imply that the functions
\[
\begin{align*}
V_A^{(n)}(\psi) &= z^{p_1} H^{(n)}(p_1, 2p_1 + 2 + \gamma, z) \quad (51) \\
V_B^{(n)}(\psi) &= z^{p_2} H^{(n)}(p_2, 2p_2 + 2 + \gamma, z), \quad (52)
\end{align*}
\]
where \( z = \frac{\delta}{\psi} \) and \( p_{1,2} \) are defined in Lemma 5, solve equations (22), (23) from section 4.

6 Boundary Conditions and Richardson Extrapolation

The results of the preceding sections calculate the general solutions of the approximating systems of ODE’s (9), (10) in the case of options on stock paying constant continuous dividends and (20), (21) in the case of Asian options based on the arithmetic mean. The problem of solving the corresponding boundary conditions has not been addressed yet. We suggest doing this part of the valuation numerically.

To be specific, we will consider the case of a European call on a stock that pays constant continuous dividends. As follows from Lemma 1 and the
results in sections 3 and 5, the general solution of the recursive sequence of non-homogeneous ODE’s (9), (10) is

$$C_1^{(n)} = \sum_{i=1}^{n} (-\eta)^{n-i} \left[ \alpha_1^{(i)} C_A^{(n-i)} + \beta_1^{(i)} C_B^{(n-i)} \right] \quad \text{for } S \in [0, K]$$

$$C_2^{(n)} = C_U^{(n)} + \sum_{i=1}^{n} (-\eta)^{n-i} \left[ \alpha_2^{(i)} C_A^{(n-i)} + \beta_2^{(i)} C_B^{(n-i)} \right] \quad \text{for } S \geq K$$

where coefficients $\alpha_1^{(i)}$, $\beta_1^{(i)}$ and $\alpha_2^{(i)}$, $\beta_2^{(i)}$ should be determined from the corresponding boundary conditions. We find these coefficients iteratively. Suppose that $\alpha_1^{(i)}$, $\beta_1^{(i)}$ and $\alpha_2^{(i)}$, $\beta_2^{(i)}$ for $i \in \{1, \cdots, n - 1\}$ are already known from the preceding iterations. At the $n^{th}$ iteration we need to determine the coefficients $\alpha_1^{(n)}$, $\beta_1^{(n)}$ and $\alpha_2^{(n)}$, $\beta_2^{(n)}$. This can be done by considering the following four conditions:

$$C_1^{(n)}(S) = C_2^{(n)}(S) \quad \text{for } S = K \quad (53)$$

$$\left(C_1^{(n)}(S)\right)_s = \left(C_2^{(n)}(S)\right)_s \quad \text{for } S = K \quad (54)$$

$$C_2^{(n)}(S) = O(S) \quad \text{as } S \to \infty \quad (55)$$

$$C_1^{(n)}(0) = \quad 0. \quad (56)$$

Conditions (53) and (54) follow from the continuity of the stock price and from the continuity of the stock price derivative. The other two conditions, (55) and (56), are implied by equations (5) and (6).

Conditions (53) and (54) can be solved numerically in a straightforward manner. Conditions (55) and (56), however, require some analytic consideration. One can check that

$$C_A^{(n)} = O\left(\frac{(\ln S)^n}{S^{p_1}}\right) \quad \text{as } S \to \infty$$

$$C_B^{(n)} = O\left(\frac{(\ln S)^n}{S^{p_2}}\right) \quad \text{as } S \to \infty.$$
for all \( n \geq 1 \). More difficulties arise in solving (56). It is impossible to obtain numerically values of \( C_1^{(n)} \) at \( S = 0 \) and at any \( S \) which is sufficiently small. The trouble comes from estimating the function \( H^{(n)} \) for very large arguments \( z = \delta \). If \( S \) is small then values of \( C_A^{(n)} \) and \( C_B^{(n)} \) overflow a computer and it looses its precision in arithmetic operations between \( C_A^{(n)} \) and \( C_B^{(n)} \).

To avoid this problem we suggest the following approach. Homogeneous equation (14) is known to have only one solution that grows polinomially as \( z \) goes to infinity\(^{16}\). This function is called Tricomi’s function. For large values of the argument \( z \) Tricomi’s function can be approximated by\(^{17}\)

\[
T^{(0)}(a, b, z) = z^{-a} \sum_{k=0}^{M_T} \frac{(a)_k (1 + a - b)_k (-1)^k}{k! z^k}.
\]

Polynomial growth of \( T^{(0)} \) as \( z \to \infty \) ensures that the function

\[
C^{(0)}_T(S) := e^{-z} z^n T^{(0)}(p + 2 - \gamma, 2p + 2 - \gamma, z), \quad z = \frac{\delta}{S}
\]

converges to zero as \( S = \downarrow 0 \). We obtained a sequence of polynomially growing functions \( T^{(n)} \) that solve recursive system (13).

**Theorem 4** The functions

\[
T^{(n)}(a, b, z) = z^{-a} \sum_{k=0}^{M_T} \frac{(a)_k (1 + a - b)_k (-1)^k}{k! z^k} \xi_k^{(n)}(a, b)
\]

where

\[
\xi_k^{(0)} = 1, \quad k \in \{0, \ldots, M_T\}
\]

\[
\xi_k^{(n)} = \sum_{i=k}^{M_T} \frac{\xi_i^{(n-1)}}{(a+i)(1+a-b+i)} \quad n \geq 1, \quad k \in \{0, \ldots, M_T\}
\]

solve the system of equation (13).

**Proof:** Apply the same arguments as in the proof of Lemma 6. \( \blacksquare \)

\(^{16}\)Other solutions grow exponentially as \( z \to \infty \).

\(^{17}\)See [1]. The integer constant \( M_T \) is a parameter of approximation.
It follows that the sequence of functions

\[ C_T^{(n)}(S) := e^{-\frac{\delta}{S}} z^n T^{(n)}(a, b, z) \quad n \geq 0, \quad z = \frac{\delta}{S} \]

solves (11), (12) and that each of these functions converges to zero as \( S \downarrow 0 \). Finally, condition (56) can be substituted by two equations

\[
\begin{align*}
\rho C_T^{(n)}(S) & = C_1^{(n)}(S) \quad \text{for } S = s_0 \\
\rho \left( C_T^{(n)}(S) \right)_s & = \left( C_1^{(n)}(S) \right)_s \quad \text{for } S = s_0
\end{align*}
\]

where \( \rho \) is an extra unknown parameter and where \( s_0 \) is any positive number less than the strike \( K \) such that \( T^{(0)} \) is a good approximation of Tricomi’s function at \( z = \frac{\delta}{s_0} \) and such that the values of \( C_A^{(n)}(s_0) \) and \( C_B^{(n)}(s_0) \) do not overflow the computer. Therefore, at the \( n \)th iteration we have five straightforward equations for five unknown parameters \( \alpha_1^{(n)}, \beta_1^{(n)}, \alpha_2^{(n)}, \beta_2^{(n)}, \) and \( \rho \).

The approximation error of this semi-analytic approach is mostly determined by \( \Delta \tau \). In each particular example, we can obtain several estimations corresponding to different \( \Delta \tau \) and then apply Richardson extrapolation to them. Figure 1 shows the values obtained by our method for one European call on a stock with constant continuous dividends. The horizontal axis corresponds to the number of time steps \( N \) that were used to subdivide the time to maturity \( T \). The lower graph shows the option’s value \( C^{(N)} \) obtained for \( \Delta \tau = \frac{T}{N} \). The upper graph corresponds to Richardson extrapolation. For each \( N \) this graph indicates the extrapolation obtained from \( C^{(1)}, \ldots, C^{(N)} \).
Figure 1: European Option.

Similar approach works for solving boundary conditions corresponding to the Asian options. Figure 2 provides data for one European style put struck at the average.
Figures 1 and 2 indicate convergence of the approximation errors to zero. Richardson extrapolation significantly improves the rate of convergence.

7 Appendix

This appendix contains some technical parts of the proof of Theorem 3. We define the coefficients\(^\text{18}\)
\[
C^{(k)}_0 := 1 \quad k \geq 1
\]
\[
C^{(k)}_m := \sum_{i_m=0}^{k-1} \sum_{i_{m-1}=0}^{i_m+1} \ldots \sum_{i_2=0}^{i_3+1} \sum_{i_1=0}^{i_2+1} 1 \quad m \geq 1, \quad k \geq 1. \quad (58)
\]

These coefficients have the following properties:

**Property 1:**
\[
C^{(1)}_{m+1} = \sum_{i=0}^{m} C^{(1)}_i C^{(1)}_{m-i} =
\]

\(^{18}\)In some sources these coefficients are referred to as Catalan's numbers.
This property provides a simple tool for calculating \( C_m^{(i)} \). One can check that \( C_0^{(1)} = 1, C_1^{(1)} = 1, C_2^{(1)} = 2, C_3^{(1)} = 5, C_4^{(1)} = 14, C_5^{(1)} = 42, \ldots \)

**Property 2:**

\[
C_m^{(k)} = C_m^{(k-1)} + C_{m-1}^{(k+1)}, \quad m \geq 1, \ k \geq 2 \tag{59}
\]

\[
C_m^{(1)} = C_{m-1}^{(2)}, \quad m \geq 1 \tag{60}
\]

**Proof of Property 1:**

\[
C_m^{(i+1)} = \sum_{i_0 = 0}^{i_m} \sum_{i_1 = 0}^{i_{m-1}} \ldots \sum_{i_{i_0} = 0}^{i_1} 1 = 
\]

\[
= \sum_{i_0 = 0}^{1} \sum_{i_1 = 0}^{i_{m-1}+1} \sum_{i_2 = 0}^{i_{m-2}+1} \ldots \sum_{i_{i_0} = 0}^{i_1+1} 1 = 
\]

\[
= C_m^{(i)} + \sum_{i_0 = 1}^{1} (C_{m-1}^{(i)} + \sum_{i_1 = 1}^{i_{m-1}+1} (+ \sum_{i_{i_0} = 1}^{i_1+1} \ldots 1)) = 
\]

\[
= C_m^{(i)} + C_{m-1}^{(i)} \begin{array}{c}
\sum_{i_0 = 1}^{1} 1 + \sum_{i_1 = 1}^{i_{m-1}+1} 1 \ldots \sum_{i_{i_0} = 1}^{i_1+1} 1
\end{array} + 
\]

\[
= C_m^{(i)} + C_{m-1}^{(i)} \begin{array}{c}
\sum_{i_0 = 0}^{0} 1 + \sum_{i_1 = 0}^{i_{m-1}+1} 1 \ldots \sum_{i_{i_0} = 0}^{i_1+1} 1
\end{array} + 
\]

\[
= C_m^{(i)} + C_{m-1}^{(i)} + C_{m-2}^{(i)} + C_{m-1}^{(i)} + C_{m-1}^{(i)} + C_m^{(i)}.
\]
PROOF OF PROPERTY 2: For $m = 1, \ k \geq 2$ we have

$$C_1^{(k)} = \sum_{i_1=0}^{k-1} 1 = \sum_{i_2=0}^{k-2} 1 + 1 = C_1^{(k-1)} + C_0^{(k+1)}.$$ 

For $m \geq 2, \ k \geq 2$

$$C_m^{(k)} = \sum_{i_m=0}^{k-1} \sum_{i_{m-1}=0}^{i_m+1} (\ldots) = \sum_{i_m=0}^{k-2} \sum_{i_{m-1}=0}^{i_m+1} (\ldots) + \sum_{i_{m-1}=0}^{((k+1)-1)} (\ldots) = C_m^{(k-1)} + C_{m-1}^{(k+1)}$$

which proves (59). We also have to show (60). Indeed, for $m = 1$

$$C_1^{(1)} = \sum_{i_1=0}^{0} 1 = 1 = C_0^{(2)}$$

and for $m \geq 2$

$$C_m^{(1)} = \sum_{i_m=0}^{0} \sum_{i_{m-1}=0}^{i_m+1} (\ldots) = \sum_{i_{m-1}=0}^{1} (\ldots) = C_{m-1}^{(2)}.$$ 

\[\blacksquare\]

PROPERTY 3: Suppose that $f_k^{(n)}, \ 0 \leq k \leq n$, satisfy equation (44)

$$f_k^{(n)} = \frac{1}{k!} \lambda^k \sum_{i=0}^{n-k} (-1)^i C_i^{(k)} \frac{1}{\lambda^{k-i}} f_0^{(n-k-i)} \quad 1 \leq k \leq n.$$ 

Then equations (46) and (47) hold. That is,

$$f_n^{(n)} = \frac{f_{n-1}^{(n-1)}}{n \lambda} \quad n \geq 1$$

$$f_k^{(n)} = \frac{f_{k-1}^{(n-1)}}{k \lambda} - \frac{(k+1)f_k^{(n)}}{\lambda} \quad 1 \leq k \leq n - 1.$$ 

PROOF OF PROPERTY 3: Equation (44) implies (46) since $C_0^{(k)} = 1$ for all $k \geq 1$ and relationship (47) follows from (44) by means of properties (59) and (60). 

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References


