14 Second-Order Propositional Logic II

14.1 Assignments

Let *Var* be the type of propositional variables, and let $\mathbb{B} = \{0, 1\}$ be the booleans (with 0 meaning false and 1 meaning true). An *assignment* is a function $v : Var \to \mathbb{B}$.

Given an assignment v, a boolean b, and a propositional variable p, the "updated" assignment $v|_b^p$ is the function (in $Var \to \mathbb{B}$) defined by

$$(v|_b^p)(q) = \begin{cases} b & \text{if } q = p \\ v(q) & \text{o.w.} \end{cases}$$

14.2 Semantics of $\mathbf{P^2}$

Let A be a $\mathbf{P^2}$-formula and let v be an assignment; let $v[A]$ be the notation for the (boolean) value of A under v, and let $v[A] : \mathbb{B}$ be defined recursively as follows:

$$\begin{array}{lcl} v[\bot] & = & 0 \\ v[p] & = & v(p) \\ v[A \supset B] & = & (\sim_{\mathbb{B}} v[A]) \vee_{\mathbb{B}} v[B] \\ v[\forall p A] & = & (v|_0^p)[A] \wedge_{\mathbb{B}} (v|_1^p)[A] \end{array}$$

where $\sim_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$, $\vee_{\mathbb{B}} : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$, and $\wedge_{\mathbb{B}} : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ are the standard boolean operators.

For a finite set of formulas Γ, define $v_\wedge[\Delta] = \bigwedge_{\mathbb{B}} \{ v[A] \mid A \in \Delta \}$ and define $v_\vee[\Gamma] = \bigvee_{\mathbb{B}} \{ v[A] \mid A \in \Gamma \}$, where $\bigwedge_{\mathbb{B}} S$ is the conjunction of the boolean values in the set S and $\bigvee_{\mathbb{B}} S$ is their disjunction. (By convention, $\bigwedge_{\mathbb{B}} \emptyset = 1$ and $\bigvee_{\mathbb{B}} \emptyset = 0$.) The value $v[\Delta \vdash \Gamma]$ of a sequent can now be defined as $(\sim_{\mathbb{B}} v_\wedge[\Delta]) \vee_{\mathbb{B}} v_\vee[\Gamma]$.

14.3 Rules of $\mathbf{P^2}$

The multiple-conclusioned sequent proof rules for $\mathbf{P^2}$ are (in "root-down tree format")

$$\Delta, \bot \vdash \Gamma$$

$$\frac{\Delta \vdash A, \Gamma \qquad \Delta, B \vdash \Gamma}{\Delta, A \supset B \vdash \Gamma} \qquad \frac{\Delta, A \vdash B, \Gamma}{\Delta \vdash A \supset B, \Gamma}$$

$$\frac{\Delta, \forall p A, A|^p_B \vdash \Gamma}{\Delta, \forall p A \vdash \Gamma} \qquad \frac{\Delta \vdash A|^p_q, \Gamma}{\Delta \vdash \forall p A, \Gamma} \text{ (!)}$$

$$\Delta, A \vdash A, \Gamma$$

$$\frac{\Delta \vdash \Gamma}{\Delta, A \vdash \Gamma} \qquad \frac{\Delta \vdash \Gamma}{\Delta \vdash A, \Gamma}$$

(!) this is only legal if $q \notin FV(\Delta, \Gamma, \forall p A)$.

The rules for $\exists$ can be derived from the rules given above:

$$\frac{\Delta, A|^p_q \vdash \Gamma}{\Delta, \exists p A \vdash \Gamma} \text{ (!)} \quad \frac{\Delta \vdash A|^p_B, \Gamma}{\Delta \vdash \exists p A, \Gamma}$$

The familiar rules for $\wedge$, $\vee$, and $\sim$ can also be derived.

an example proof:

$$\cfrac{\cfrac{\bot \vdash \bot}{\forall p.p \vdash \bot}}{\vdash (\forall p.p) \supset \bot}$$

The topmost step is the left $\forall$ rule, using $\bot$ as B; i.e., informally, the proof is

$$\cfrac{\cfrac{\cfrac{\bot \vdash \bot}{p|^p_\bot \vdash \bot}}{\forall p.p \vdash \bot}}{\vdash (\forall p.p) \supset \bot}$$

where the topmost pseudo-step is justified (meta-theoretically) by the equality $p|^p_\bot = \bot$.

14.4 First-Order Logic

This is the calculus one usually has in mind when using the word “logic”. It is expressive enough for all of mathematics, except for those concepts that rely on a notion of construction or computation. However, dealing with more advanced concepts is often somewhat awkward and researchers often design specialized logics for that reason.

Our account of first-order logic will proceed similar to the one of propositional logic. We will present

- The *syntax*, or the formal language of first-order logic, that is symbols, formulas, subformulas, formation trees, substitution, etc.
- The *semantics* of first-order logic
- *Proof systems* for first-order logic, such as the axioms, rules, and proof strategies of the first-order tableau method and refinement logic
- The *meta-mathematics* of first-order logic, which established the relation between the semantics and a proof system

In many ways, the account of first-order logic is a straightforward extension of propositional logic. One must, however, be aware that there are subtle differences.

— see handwritten notes for further details —