13 Second-Order Propositional Logic

13.1 Motivation

— see handwritten notes —

13.2 Syntax of $\mathbf{P^2}$

In the following exposition we will use the following symbols as meta-variables

$p, q, r, \ldots$	a propositional variable
$A, B, \ldots$	a $\mathbf{P^2}$ formula
$\Gamma, \Delta, \ldots$	a finite set of $\mathbf{P^2}$ formulas

13.2.1 Formulas of $\mathbf{P^2}$

The formulas of $\mathbf{P^2}$ are generated by

$$\begin{array}{lcl} V & \rightarrow & p_0 \mid p_1 \mid p_2 \mid \cdots \mid p_i \mid \cdots \qquad \text{(a countably infinite set)} \\ A & \rightarrow & V \mid \bot \mid (A \supset A') \mid (\forall V\, A) \end{array}$$

Examples: $(p_0 \supset p_1)$, $(\forall p_0 (p_0 \supset p_1))$, $(\forall p_1((\forall p_2(p_2 \supset p_2)) \supset \bot))$

Intuitively, the meaning of $\mathbf{P^2}$ formulas is obvious.

13.2.2 Increased Expressiveness

A formula like $((\forall p_0\, p_0) \supset \bot)$ states that it is impossible to make every propositional formula true. Statements of this nature could not be expressed in ordinary propositional logic.

13.2.3 Defined Connectives

The remaining connectives and quantifier can be defined in terms of $\bot$, $\supset$, and $\forall$:

$$\begin{array}{rcl} \sim A & \leftrightarrow & A \supset \bot \\ A \wedge B & \leftrightarrow & \sim(A \supset \sim B) \\ A \vee B & \leftrightarrow & (\sim A) \supset B \\ \exists p A & \leftrightarrow & \sim(\forall p \sim A) \end{array}$$

13.3 Substitution

Substitution is the key to formal reasoning about quantified formulas. In order to explain it, we need to understand the role of variable occurrences in a formula.

13.3.1 Free and Bound Variables

Quantified variables are considered to be *bound* in the formula that begins with the corresponding quantifier. Otherwise they are considered to be *free*. Free variables stand for arbitary propositional formulas, which means that the truth of the formula should not change if the variable is instantiated.

For A a formula of $\mathbf{P^2}$, the set of propositional variables that are free in A, denoted $FV(A)$, can be characterized by the following recursive definition:

$$\begin{array}{lcl} FV(\bot) & = & \emptyset \\ FV(p) & = & \{p\} \\ FV(A\supset B) & = & FV(A)\cup FV(B) \\ FV(\forall pA) & = & FV(A)-\{p\} \end{array}$$

The set of all propositional variables that occur in A, $PV(A)$, can likewise be defined as

$$\begin{array}{lcl} PV(\bot) & = & \emptyset \\ PV(p) & = & \{p\} \\ PV(A\supset B) & = & PV(A)\cup PV(B) \\ PV(\forall pA) & = & PV(A)\cup\{p\} \end{array}$$

Examples:

$$\begin{array}{lcc} FV(p_0\supset p_1) & = & \{p_0,p_1\} \\ PV(p_0\supset p_1) & = & \{p_0,p_1\} \\ FV(\forall p_0(p_0\supset p_1)) & = & \{p_1\} \\ PV(\forall p_0(p_0\supset p_1)) & = & \{p_0,p_1\} \\ FV(\forall p_1((\forall p_2(p_2\supset p_2))\supset\bot)) & = & \emptyset \\ PV(\forall p_1((\forall p_2(p_2\supset p_2))\supset(\forall p_3 p_1))) & = & \{p_1,p_2,p_3\} \end{array}$$

We can extend the definitions of FV and PV to finite sets of formulas by taking $FV(\Gamma)=\bigcup_{A\in\Gamma}FV(A)$ and likewise by taking $PV(\Gamma)=\bigcup_{A\in\Gamma}PV(A)$.
For sequents, the definitions are $FV(\Delta\vdash\Gamma)=FV(\Delta\cup\Gamma)$ and $PV(\Delta\vdash\Gamma)=PV(\Delta\cup\Gamma)$.

13.4 Defining Substitution

Substitution $A|_B^p$ is the replacement of *all* occurrences of the variable p in A by the formula B. There are a few issues, however, that one needs to be aware of.

Variables that are bound by a quantifier, must not be replaced, as this would change the meaning. $(\exists p.p\supset\sim q)|_q^p$ should not result in $(\exists p.q\supset\sim q)$ as the former is a tautology (choose $p=\bot$) while the latter depends on the value of q (and ths is only satisfiable).

In the same way, a variable must not be replaced by a bound variable, as this may change the meaning of the formula. For instance, the formula $\exists q((p\supset q)\wedge(q\supset p))$ is a tautology (choose $q=p$), but defining $\exists q((p\supset q)\wedge(q\supset p))|_{\sim q}^p$ as $\exists q((\sim q\supset q)\wedge(q\supset\sim q)$ is unsatisfiable.

The formal definition takes both issues into account. In the former case, nothing will be substitutet, in the latter case, variable *capture* is avoided by renaming the bound variable first.

Given formulas A and B of $\mathbf{P^2}$ and a propositional variable p, the $\mathbf{P^2}$ formula $A|^p_B$ ("A with B substituted for p") is, as usual, defined recursively:

$$
\begin{array}{rcll}
\bot|^p_B & = & \bot & \\
p|^p_B & = & B & \\
q|^p_B & = & q & (q \neq p) \\
(A{\supset}A')|^p_B & = & (A|^p_B){\supset}(A'|^p_B) & \\
(\forall p A)|^p_B & = & \forall p A & \\
(\forall q A)|^p_B & = & \forall q (A|^p_B) & (q \neq p,\ q \notin FV(B)) \\
(\forall q A)|^p_B & = & \forall q' (A|^q_{q'}|^p_B) & (q \neq p,\ q \in FV(B),\ q' \notin PV(A, B, p))
\end{array}
$$

Examples:

$$
\begin{array}{rcl}
(p_0{\supset}p_1)|^{p_0}_{p_2\supset p_3} & = & ((p_2{\supset}p_3){\supset}p_1) \\
(p_0{\supset}(p_0{\supset}p_1))|^{p_0}_{p_3} & = & (p_3{\supset}(p_3{\supset}p_1)) \\
(p_0{\supset}p_0)|^{p_0}_{p_0\supset p_0} & = & ((p_0{\supset}p_0){\supset}(p_0{\supset}p_0)) \\
(p_0{\supset}(\forall p_0(p_0{\supset}p_0)))|^{p_0}_{p_1} & = & (p_1{\supset}(\forall p_0(p_0{\supset}p_0))) \\
(\forall p_0(p_0{\supset}p_3))|^{p_3}_{p_0} & = & (\forall p_1(p_1{\supset}p_0))
\end{array}
$$

Again one can extend substitution to finite sets of formulas and thence to sequents by letting $\Gamma|^p_B = \{\, A|^p_B \mid A \in \Gamma \,\}$ and $(\Delta \vdash \Gamma)|^p_B = (\Delta|^p_B) \vdash (\Gamma|^p_B)$.