# A Constructively Adequate Refutation System for Intuitionistic Logic

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**Abstract.** We present a refutation system for intuitionistic logic which allows to locate situations where constructive reasoning is unnecessary. In many subgoals of a derivation this property makes it possible to reduce the set of relevant reduction rules to a confluent one. As a result, backtracking over choicepoints for different orderings of non-permutable rules can often be avoided and the search space will be reduced considerably.

## 1 Introduction

Traditional calculi for intuitionistic logic [3, 4] are strictly constructive in the sense that they only allow constructive reasoning even if it is clear that a classical proof will lead to the same result. While this property may be very useful for pedagogical purposes it turns out to be a real handicap for automated proof search procedures, particularly for those based on connecting complementary literals such as matrix methods [12, 17], resolution type provers [16], or translation methods [10, 11]. Since these methods were derived from the traditional calculi they have to carry around the "constructive overhead" even if it does not contribute to the essential argument of the proof. In many situations this leads to an unnecessary explosion of the search space. Furthermore, many decidable subclasses of intuitionistic logic (such as intuitionistic propositional logic) can only be decided by specialized methods [5, 13] while the existing general proof procedures fail to detect invalid theorems.

Recently developed calculi [2, 6, 9] are more adequate in the sense that they reduce the number of unnecessary constructive arguments within an intuitionistic proof. In these calculi, however, the theoretical insights are used in a rather hidden and unintuitive manner, which makes it difficult to use them as a foundation for automated proof search. Furthermore they introduce additional non-permutabilities which may again lead to a blow-up of the induced search space.

In this paper we present a refutation system for intuitionistic logic which allows to locate situations where constructive reasoning is unnecessary and is also well suited for automation. The general design decision was to integrate a mechanism controlling the set of assumptions about already existing constructions into the

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calculus. As a result it will be possible to determine if a new assumption follows from the already existing ones *before* it will be introduced by a rule of the calculus. If this is the case, then constructive and non-constructive reasoning will lead to the same results in the current branch of the proof and we may continue classically. This methodology has the following advantages:

– *Search Space Reduction*: In many branches of the proof the intuitionistic search space will be identical to the classical one. In particular, the search space for formulas from intuitionistic propositional logic and many other decidable subclasses of intuitionistic logic will become finite and a proof search method will be able to decide their validity.
– *Adequacy*: Strictly constructive reasoning will not be performed if we can detect that a certain construction does not contribute to the evidence which is necessary for proving the theorem.
– *Automation*: Because of its formal similarity to Fitting's calculi [3] our calculus can be used as foundation for the development of improved matrix-methods in the style of Wallen [17] or improved translations into classical logic [11] which then will lead to more efficient proof procedures for intuitionistic logic

Our paper is organized as follows. In section 2 we introduce some basic definitions concerning the Kripke-style semantics of intuitionistic logic. Section 3 describes our formal calculus and its relation to the semantical notion of knowledge accumulation. In section 4 we discuss soundness and completeness of our calculus, show that propositional formulas can be decided within our calculus, and explore the connection to classical logic and the resulting reduction of the search space. We conclude with a survey of related work and future prospects.

## 2 Preliminaries

In this section we introduce some basic definitions which we will refer to throughout the rest of this paper. We assume pairwisely disjoint alphabets $\mathcal{V}, \mathcal{C}, \mathcal{P}^n$ of variable symbols, parameters and $n$-ary predicate symbols respectively. (First-order) formulas over these alphabets and the logical connectives $\wedge, \vee, \Rightarrow, \neg, \forall, \exists$ are defined as usual. By *atomic* formulas we mean formulas without logical connectives. A *ground formula* is a formula without occurrences of free variables. If not stated otherwise we assume every formula to be ground.

The following definitions of intuitionistic interpretation, forcing, model and validity combine the respective definitions in [3] and [14].

**Definition** **1** *Intuitionistic interpretation*

Let $\mathbf{W}$ be a non–empty set, $\mathbf{R} \subseteq \mathbf{W} \times \mathbf{W}$ a transitive and reflexive relation (*accessibility*), $\mathbf{D}$ a mapping from $\mathbf{W}$ to non–empty subsets of $\mathcal{C}$, and $\mathbf{V}$ an *evaluation function* mapping elements of $\mathbf{W}$ to sets of atomic ground formulas.

Then $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{V} \rangle$ is said to be an *intuitionistic interpretation* iff $\mathbf{D}(w) \subseteq \mathbf{D}(v)$, for any $w, v \in \mathbf{W}$ with $w\mathbf{R}v$. □

For a given $I_j = \langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{V} \rangle$ and a given $w \in \mathbf{W}$ we denote by $w^*$ an arbitrary $v \in \mathbf{W}$ with $w\mathbf{R}v$. Thus by saying "*$P$ holds for all $w^*$*" we mean "*$P$ holds for all $v$ with $w\mathbf{R}v$*" and "*there is a $w^*$ such that $P$*" means "*there is a $v$ with $w\mathbf{R}v$ such that $P$*". By $\widehat{\mathbf{D}}(w)$ we denote all first-order formulas which can be constructed using only the parameters in $\mathbf{D}(w)$.

**Definition** **2** *Intuitionistic forcing, model, validity*

Let $I_j = \langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{V} \rangle$ be an intuitionistic interpretation, $w \in \mathbf{W}$.

A formula $A$ is said to be *forced* (otherwise *unforced*) at $w$ — denoted as $w \Vdash A$ (and $w \nVdash A$ respectively) — iff one of the following conditions holds:

1. $A$ is atomic, $A \in \widehat{\mathbf{D}}(w)$, and $A \in \mathbf{V}(w^*)$ for any $w^*$
2. $A = A_1 \wedge A_2$, $w \Vdash A_1$, and $w \Vdash A_2$
3. $A = A_1 \vee A_2$, $A_1 \vee A_2 \in \widehat{\mathbf{D}}(w)$, and $w \Vdash A_1$ or $w \Vdash A_2$
4. $A = \neg A_1$, $\neg A_1 \in \widehat{\mathbf{D}}(w)$, and $I_j(w^*) \nVdash A_1$ for any $w^*$
5. $A = A_1 \Rightarrow A_2$, $A_1 \Rightarrow A_2 \in \widehat{\mathbf{D}}(w)$, and $I_j(w^*) \Vdash A_2$ if $I_j(w^*) \Vdash A_1$ for any $w^*$
6. $A = \forall x.A_1$ and $I_j(w^*) \Vdash A_1[x \backslash c]$ for any $w^*$ and any $c \in \mathbf{D}(w^*)$
7. $A = \exists x.A_1$ and $w \Vdash A_1[x \backslash c]$ for some $c \in \mathbf{D}(w)$

$I_j$ is said to be an *intuitionistic model* (otherwise *countermodel*) for $A$ — denoted as $I_j \models_j A$ ($I_j \not\models_j A$ respectively) — iff $w \Vdash A$ for any $w \in \mathbf{W}$ with $A \in \widehat{\mathbf{D}}(w)$. $A$ is said to be *intuitionistically valid* (otherwise *invalid*) — denoted as $\models_j A$ ($\not\models_j A$ respectively) — iff every interpretation is a model for it. □

Throughout this paper we will often refer to the intuitionistic Beth-tableaux system (see e.g. [3]) which we will denote by $\mathsf{TJ}$. The corresponding classical tableaux system (see e.g. [15]) will be called $\mathsf{TC}$ and classical validity will be denoted by $\models_c$ (cf. [1]). The following correlation between forcing and accessibility will be frequently referred to as the *heredity condition* or *heredity lemma*:

**Lemma** **1** *Heredity of forcing*

Let $I_j = \langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{V} \rangle$ be an intuitionistic interpretation and $A$ be a formula.

Then for any $w \in \mathbf{W}$: $w \Vdash A \;\Rightarrow\; w^* \Vdash A$ □

The proof is a straightforward induction over the degree of $A$ (see [3] for details).

Basically, forcing is used to reflect the concept of *known evidence* for a formula. If $A$ is forced at some $w \in \mathbf{W}$ then it is known at $w$ how to construct evidence for the validity of $A$. Consequently, the heredity lemma states the availability of this knowledge at every $w^*$. Therefore the elements of $\mathbf{W}$ can be considered as *knowledge stages* describing the information about constructions which has been gathered so far.

## 3 The calculus

In this section we introduce the formal definition of our refutation system $\mathsf{TJ}_A$ and illustrate the principal ideas on which it is based. To this end we shall discuss the correlation between forcing and the semantical notion of knowledge acquisition which eventually led to the design of the refutation system.

### 3.1 The formal system

As for $\mathsf{TJ}$ we use a notion of *signed formulas*. However, in addition to the signs $\mathbf{T}$ and $\mathbf{F}$ we introduce a third sign $\shortmid\!\mathbf{T}$. Its purpose will be explained later.

**Definition** **3** *Signed formula*
Let $A$ be a formula. Then $\mathbf{T}A$, $\shortmid\!\mathbf{T}A$ and $\mathbf{F}A$ are *signed formulas*. □

The next definition extends the definition of rules in [3] by a concept of *prefixes*:

**Definition** **4** *Prefixed set of signed formulas, reduction rule*
Let $P$ be a list of formulas and $S$ be a set of signed formulas. Then the construct $P\colon\ S$ is called a *prefixed set of signed formulas* (*PSSF*) where $P$ is the *prefix* and $S$ is the *suffix*. If $F \notin S$ is a signed formula then $S, F$ denotes the set $S \cup \{F\}$. If $P = [A_1, \ldots, A_n]$ is a prefix then $X.P$ denotes the prefix $[X, A_1, \ldots, A_n]$.
A *reduction rule* is denoted as

$$\frac{P_1\colon\ S_1 \quad [P_2\colon\ S_2]}{P\colon\ S, F}\ \mathit{name}$$

where *name* is the name, $P\colon\ S, F$ is the *conclusion* and $P_1\colon\ S_1$ as well as (possibly) $P_2\colon\ S_2$ are the *premises* of the rule. A constraint $A$"$\in$"$P$ added to the name of a reduction rule denotes that the rule is only applicable if $\neg A$ or $A \Rightarrow X$ for an arbitrary formula $X$ is an element of the prefix. By adding $P = \neg X.P'$ we indicate that a rule is applicable only if the first element of the prefix is a negated formula. $S_{\mathbf{T}}$ denotes the set $\{\mathbf{T}A \mid \mathbf{T}A \in S \text{ or } \shortmid\!\mathbf{T}A \in S\}$. □

By an *application* of a rule $R$ to a PSSF $P\colon\ S$ we mean the replacement of $P\colon\ S$ by $P_1\colon\ S_1$ (and possibly $P_2\colon\ S_2$) where $P\colon\ S$ is an instantiation of the conclusion of $R$ and $P_1\colon\ S_1$ (and $P_2\colon\ S_2$) are corresponding instantiations of the premise(s) of $R$. By a *configuration* we mean a finite collection of PSSFs. An application of $R$ to a configuration $C = \{P_1\colon\ S_1, \ldots, P_n\colon\ S_n\}$ generates a new configuration $C'$ by replacing one of the $P_i\colon\ S_i$ by the result(s) of applying $R$ to it. A *tableau* is a finite sequence of configurations such that each configuration (except of the first) results from an application of some rule $R$ to its immediate predecessor. A tableau for a PSSF $P\colon\ S$ is a tableau starting with $\{P\colon\ S\}$.
A PSSF $P\colon\ S$ is called *closed* if $\mathbf{F}A, \mathbf{T}A \in S$ or $\mathbf{F}A, \shortmid\!\mathbf{T}A \in S$ for an arbitrary formula $A$. A configuration is closed if it contains only closed PSSFs. A tableau is closed if it contains a closed configuration. A PSSF with finite suffix and prefix is *inconsistent* if there is a closed tableau for it. Otherwise it is *consistent*. A PSSF $P\colon\ S$ with finite prefix $P$ and infinite suffix $S$ is consistent if $P : S'$ is consistent for every finite subset $S'$ of $S$. A formula $A$ is a *theorem* if $: \mathbf{F}A$ is inconsistent. A closed tableau for $: \mathbf{F}A$ is called a proof of $A$.
Based on these definitions we define our refutation system $\mathsf{TJ}_A$ as the set of reduction rules which are displayed in fig. 1.

$$\frac{P\colon\ S,\,\mathbf{F}A \qquad P\colon\ S,\,\mathbf{F}B}{P\colon\ S,\,\mathbf{F}A\wedge B}\ \mathbf{F}\text{-}\wedge \qquad\qquad \frac{P\colon\ S,\,\mathbf{T}A,\,\mathbf{T}B}{P\colon\ S,\,\mathbf{T}A\wedge B}\ \mathbf{T}\text{-}\wedge$$

$$\frac{P\colon\ S,\,\mathbf{F}A,\,\mathbf{F}B}{P\colon\ S,\,\mathbf{F}A\vee B}\ \mathbf{F}\text{-}\vee \qquad\qquad \frac{P\colon\ S,\,\mathbf{T}A \qquad P\colon\ S,\,\mathbf{T}B}{P\colon\ S,\,\mathbf{T}A\vee B}\ \mathbf{T}\text{-}\vee$$

$$\frac{P\colon\ S,\,\mathbf{T}A}{P\colon\ S,\,\mathbf{F}\neg A}\ \mathbf{F}\text{-}\neg\text{-}1\ \ (P=\neg X.P') \qquad\qquad \frac{P\colon\ S,\,\imath\mathbf{T}\neg A,\,\mathbf{F}A}{P\colon\ S,\,\mathbf{T}\neg A}\ \mathbf{T}\text{-}\neg$$

$$\frac{\neg A.P\colon\ S_{\mathbf{T}},\,\mathbf{T}A}{P\colon\ S,\,\mathbf{F}\neg A}\ \mathbf{F}\text{-}\neg\text{-}2\ \ (A\text{“}\notin\text{”}P,\ P\neq\neg X.P')$$

$$\frac{P\colon\ S,\,\mathbf{F}B}{P\colon\ S,\,\mathbf{F}A\Rightarrow B}\ \mathbf{F}\text{-}\Rightarrow\text{-}0\ \ (A\text{“}\in\text{”}P,\ P\neq\neg X.P')$$

$$\frac{P\colon\ S,\,\mathbf{T}A,\,\mathbf{F}B}{P\colon\ S,\,\mathbf{F}A\Rightarrow B}\ \mathbf{F}\text{-}\Rightarrow\text{-}1\ \ (P=\neg X.P') \qquad \frac{P\colon\ S,\,\imath\mathbf{T}A\Rightarrow B,\,\mathbf{F}A \qquad P\colon\ S,\,\mathbf{T}B}{P\colon\ S,\,\mathbf{T}A\Rightarrow B}\ \mathbf{T}\text{-}\Rightarrow$$

$$\frac{A\Rightarrow B.P\colon\ S_{\mathbf{T}},\,\mathbf{T}A,\,\mathbf{F}B}{P\colon\ S,\,\mathbf{F}A\Rightarrow B}\ \mathbf{F}\text{-}\Rightarrow\text{-}2\ \ (A\text{“}\notin\text{”}P,\ P\neq\neg X.P')$$

$$\frac{P\colon\ S,\,\mathbf{F}\exists x.\,A,\,\mathbf{F}A[x\backslash c]}{P\colon\ S,\,\mathbf{F}\exists x.\,A}\ \mathbf{F}\text{-}\exists \qquad\qquad \frac{P\colon\ S,\,\mathbf{T}\ A[x\backslash a]}{P\colon\ S,\,\mathbf{T}\exists x.\,A}\ \mathbf{T}\text{-}\exists\quad (a\ \text{new})^{3}$$

$$\frac{(\forall x.\,A).P\colon\ S_{\mathbf{T}},\,\mathbf{F}A[x\backslash a]}{P\colon\ S,\,\mathbf{F}\forall x.\,A}\ \mathbf{F}\text{-}\forall\quad (a\ \text{new})^{1} \qquad\qquad \frac{P\colon\ S,\,\mathbf{T}\forall x.\,A,\,\mathbf{T}A[x\backslash c]}{P\colon\ S,\,\mathbf{T}\forall x.\,A}\ \mathbf{T}\text{-}\forall$$

**Fig. 1:** Reduction rules

### 3.2 Reduction rules and knowledge accumulation

In order to illustrate the ideas underlying our formal system we shall now explore the correspondence between the behavior of the reduction rules and the semantical notion of knowledge accumulation.

The main motivation for manipulating PSSFs is to represent assumptions about a current knowledge stage $w \in \mathbf{W}$ *within* our calculus. Every **T**-formula and every $\imath\mathbf{T}$-formula in the suffix is assumed to be forced at some $w \in \mathbf{W}$. In other words, at knowledge stage $w$ it is assumed to be known how to construct evidence for the formula. At the same stage every **F**-formula in the suffix is assumed to be unforced (i.e. evidence is assumed to be definitely unknown at $w$). Applying inference rules from the conclusion to the premises (i.e. stating subgoals that remain to be proved) may correspond to advancing to another accessible knowledge stage, thus gathering additional knowledge. The prefix basically represents a chronologically ordered collection of all the evidences which have been constructed earlier during a derivation. From this "constructive history" we will then be able to derive whether an inference actually introduces new constructive evidence. If we can justify that this is not the case then we can replace such an

[1] "*a new*" expresses that $a \in \mathcal{C}$ and that $a$ must not occur in the conclusion.

inference by a non-constructive one. Therefore some of the rules of $\mathsf{TJ}_A$ have to differ from the corresponding $\mathsf{TJ}$-rules in a way which we will explain in the rest of this section.

**The rules F-⇒-2 and F-¬-2:** For implicative formulas the forcing condition given in def. 2 can be read as follows: if our current knowledge allows us to predict that we will know a construction for $B$ whenever we have acquired one for $A$ then we must know how to construct $B$ from $A$ and hence possess an evidence for $A \Rightarrow B$. Conversely, if we can extend our current knowledge by a construction for $A$ without obtaining one for $B$ then we cannot possess evidence for $A \Rightarrow B$. This insight is reflected by the **F**-⇒-2 rule which is equivalent to the **F**-⇒ rule in $\mathsf{TJ}$ if the prefix is ignored.

The conclusion $S, \mathbf{F}A \Rightarrow B$ describes a stage at which constructions for all **T**-formulas (and ı**T**-formulas) in $S$ are known while no construction for any of the **F**-formulas in $S$ or for $A \Rightarrow B$ is known. It can be inferred from the premise that we do not know how to obtain a construction for $B$ (i.e. $\mathbf{F}B$) even if we add an evidence for $A$ (i.e. $\mathbf{T}A$) to the set $S_{\mathbf{T}}$ of currently known constructions. Note that in the premise we cannot assume any of the unknown constructions of the conclusion (i.e. the previous knowledge stage) to remain unknown after introducing the evidence for $A$. Therefore the rule must remove any assumption about unknown constructions from $S$ and leave only the assumption that $B$ is unforced. Since in general the excluded assumptions cannot be regained, an application of **F**-⇒-2 is not invertible. Observe that **F**-⇒-2 extends the prefix in order to denote that by adding evidence for $A$ we may have reached a knowledge stage which contains more information than the present one.

**F**-¬-2 is similar to **F**-⇒-2 since negation is strongly related to implication. These two rules constitute the typical behavior of implication and negation while the other **F**-¬-/**F**-⇒-rules describe special cases.

**The rule F-⇒-0:** This rule handles the situation where either $A \Rightarrow X$ or $\neg A$ occurs in the current prefix. This means that **F**-⇒-2 or **F**-¬-2 must have been applied previously in the tableau construction since we always begin with an empty prefix. Thus $\mathbf{T}A$ has already been added to the respective PSSF which semantically means that evidence for $A$ has been constructed at some previous knowledge stage $w$. By the heredity lemma this construction is still known at the current stage. Thus in order to conclude that we cannot construct evidence for $A \Rightarrow B$ it suffices to show that we do not have evidence for $B$ while we do not need again to assume a construction for $A$ throughout the rest of the derivation. This also allows for keeping the assumptions about other unknown constructions.

A corresponding **F**-¬-0 rule would be meaningless since it would not add any information to the conclusion but only remove **F**-$\neg A$.

**The rules F-⇒-1 and F-¬-1:** The above considerations have demonstrated that a rule does not have to add a **T**-formula to a conclusion if we can semantically justify that the corresponding evidence is already part of our knowledge. In such cases, constructions for these formulas are always available in the remaining proof so we do not need to reason constructively about them anymore.

In our justification of **F**-⇒-0 we have determined such situations by *explicitly* tracing already introduced constructions. The rules **F**-⇒-1 and **F**-¬-1, however, rely on *implicit* assumptions about known constructions, namely on the *maximality* of an actual knowledge stage w.r.t. a given set of formulas $\Phi$. Informally, this means that we cannot gain any evidence for currently unknown formulas from $\Phi$ by advancing to another knowledge stage:

**Definition** **5** *Maximal knowledge stage*

Let $I_j = \langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{V} \rangle$ be an intuitionistic interpretation and $\Phi$ be a finite set of formulas. A knowledge stage $w \in \mathbf{W}$ is said to be *$\Phi$-maximal* iff for any $A \in \Phi$ the following proposition is true:

$$(\exists w^*.w^* \Vdash A) \;\Rightarrow\; w \Vdash A \qquad \square$$

In other words, $w$ is maximal w.r.t. $\Phi$ if for every formula $A$ in $\Phi$ either evidence can be constructed at stage $w$ or it will be impossible to construct it at all. Hence at $w$ the epistemological notion of *knowing a construction* for $A$ is virtually equivalent to the ontological notion of *truth* and classical reasoning will be sufficient for proving $A$ to be valid or invalid. The following theorem shows that for a finite set of formulas such a maximal knowledge stage must always exist.

**Theorem** **1** *Existence of maximal knowledge stage*

Let $I_j = \langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{V} \rangle$ be an intuitionistic interpretation, $w \in \mathbf{W}$ and $\Phi$ be a finite set of formulas. Then there exists a $\Phi$-maximal $w^*$.

**Proof:** Let $\Phi = \{A_1, \ldots, A_n\}$. If there is no $w^*$ forcing an $A_i$ which is unforced at $w$ then $w$ is $\Phi$-maximal. Otherwise consider some $w^*$ where $A_i$ is forced. Then $A_i$ is also forced at any $w^{**}$ by lemma 1 so we only need to look for some $A_j \in \Phi \setminus \{A_i\}$ to become forced at some $w^{**}$. If there is no such $w^{**}$ then $w^*$ is $\Phi$-maximal. Otherwise we proceed in the same manner for $w^{**}$ and $\Phi \setminus \{A_i, A_j\}$. Obviously this process terminates after at most $k$ steps at some $\Phi$-maximal $w^{(*^k)}$ for $k \in \{0, \ldots, n\}$. By transitivity of $\mathbf{R}$ $w^{(*^k)}$ is also a $w^*$. $\square$

**Corollary** **1** *Persistence of maximality*

If $w$ is $\Phi$-maximal so is any $w^*$. $\square$

Proof-theoretically, if a PSSF $P\colon\ S$ is supposed to represent assumptions about a $\Phi$-maximal knowledge stage then no inference will actually add new evidence for a formula of $\Phi$. Thus any $\mathbf{F}A \in S$ with $A \in \Phi$ does not have to be dropped by any rule application since it is sure that $A$ will never become known at a later stage. This situation will become particularly interesting if we consider the set of all the subformulas of a PSSF.

**Definition** **6** *Subformula set*
Let $P\colon\ S$ be a PSSF with $S = \{F_1, \dots F_n\}$ where each $F_i$ is of the form $F_i = \circ A_i$ with $\circ \in \{\mathbf{T}, \mathbf{\mathsf{I}T}, \mathbf{F}\}$. Then $\mathcal{G}(P\colon\ S)$ is defined as the (finite) set of all parameters occurring in some $A \in P$ as well as some $A_i$ and $\mathcal{S}(\mathcal{G}(P\colon\ S))$ is the (finite) set of all subformulas of every $A_i$ which can be constructed using only the parameters in $\mathcal{G}(P\colon\ S)$. $\mathcal{S}(\mathcal{G}(P\colon\ S))$ is called the *subformula set* for $P\colon\ S$. □

If we could justify that a given PSSF $P\colon\ S$ represents assumptions about a $\mathcal{S}(\mathcal{G}(P\colon\ S))$-maximal knowledge stage then we were allowed to apply from now on any classical rule which only introduces formulas from the current subformula set. Such a situation will occur if $P\colon\ S$ does not contain **F**-formulas. In this case all the signed formulas in $P\colon\ S$ represent assumptions about known constructions. By the heredity condition this knowledge remains available if we advance to some $\mathcal{S}(\mathcal{G}(P\colon\ S))$-maximal knowledge stage, which by theorem 1 must exist. Thus we may reason about $P\colon\ S$ with suitable classical inference rules.

Since the only rule which provides such a situation is **F**-$\neg$-2, our calculus must ensure that classical reasoning will only be applied to the premise of **F**-$\neg$-2 and any PSSF resulting from such an application. For this reason we have designed **F**-$\neg$-2 to extend the prefix in the premise by a negated formula. This causes **F**-$\Rightarrow$-2 and **F**-$\neg$-2, which have a strongly constructive behavior, to become inapplicable. Instead, we have to apply **F**-$\Rightarrow$-1 and **F**-$\neg$-1 if we want to deal with implicative or negated **F**-formulas. These rules correspond to the classical TC-rules and do not change the prefix. Since the behavior of all the other rules (except for **F**-$\forall$) is identical to their TC-counterparts we will in fact perform classical reasoning as soon as a negated formula occurs at the beginning of a prefix, i.e. after an application of **F**-$\neg$-2. Thus by a very simple syntactical concept our calculus achieves classical reasoning at situations where the task virtually is to show the impossibility of a construction rather than its existence.

As mentioned above, however, the application of classical rules must be restricted to those rules which introduce only formulas from $\mathcal{S}(\mathcal{G}(P\colon\ S))$. This is obvious for the propositional rules and we shall show now, why even the first-order inference rules except for **F**-$\forall$ do not essentially violate the $\mathcal{S}(\mathcal{G}(P\colon\ S))$-maximality.

**The rules F-$\exists$, T-$\exists$ and T-$\forall$:** For investigating **F**-$\exists$ let us assume that the prefix $P$ of a conclusion $P\colon\ S, \mathbf{F}\exists x.A$ has the form $P = \neg X.P'$ which, according to the above considerations, means that $P\colon\ S, \mathbf{F}\exists x.A$ represents assumptions about a $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\exists x.A))$-maximal knowledge stage $w$. The premise of **F**-$\exists$, however, contains the new assumption that a construction for $A[x\backslash c]$ is *not* known and $A[x\backslash c]$ is not necessarily contained in $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\exists x.A))$. The following argument shows that nevertheless $A[x\backslash c]$ cannot be forced at $w$ and may therefore be introduced without requiring us to continue constructively again.

Theorem 1 ensures the existence of a $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\exists x.A, \mathbf{F}A[x\backslash c]))$-maximal knowledge stage $w^*$ which, by corollary 1, is also $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\exists x.A))$-maximal. Because of $\exists x.A \in \mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\exists x.A))$ we know that $w \not\Vdash \exists x.A$ and therefore

$w^* \not\Vdash \exists x.A$ by the definition of maximality. According to the definition of forcing we may conclude $w^* \not\Vdash A[x\backslash c]$ and by heredity $w \not\Vdash A[x\backslash c]$.

For the applicability of $\mathbf{T}$-$\exists$ and $\mathbf{T}$-$\forall$ we can use similar arguments.

**The rule F-∀:** Within the terminology of knowledge stages the forcing condition for universally quantified formulas in def. 2 can be read as follows: if our current knowledge allows us to construct evidence for $A$ for arbitrary instances of the parameter $x$ which may not even exist at present, then we must possess a universal construction method for $A$ and hence for $\forall x.A$. Conversely, if we can extend our knowledge by constructing an instance $a$ of $x$ for which $A$ has no evidence then we cannot possess a construction for $\forall x.A$. This insight is expressed by our $\mathbf{F}$-$\forall$ rule which corresponds to the $\mathbf{F}$-$\forall$ rule in $\mathsf{TJ}$.

As usual, the conclusion $S, \mathbf{F}\forall x.A$ describes a knowledge stage at which constructions for all $\mathbf{T}$- and $|\mathbf{T}$-formulas in $S$ are known while there is no evidence for $\forall x.A$ and any of the $\mathbf{F}$-formulas. It can be inferred from a premise describing a stage where the presently known constructions do not provide evidence for $A[x\backslash a]$. Since we do not yet know anything about $a$ it must be new. After adding its construction to our knowledge some previously unknown construction may not remain unknown. Therefore we must drop all assumptions about unknown constructions other than $A[x\backslash a]$.

The extension of the prefix indicates that a value $a$ for $x$ was constructed. Since $a$ must be new w.r.t. the current PSSF its construction is unknown even if $\mathbf{F}$-$\forall$ has previously been applied, i.e. if $\forall x.A$ occurs in the prefix and we cannot optimize $\mathbf{F}$-$\forall$ as in the case of the $\mathbf{F}$-$\Rightarrow$-0 rule.

Moreover, we cannot apply a classical version of $\mathbf{F}$-$\forall$ if the conclusion describes a $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\forall x.A))$-maximal knowledge stage $w$, i.e. if $P$ has the form $\neg X.\, P'$. Although theorem 1 ensures the existence of a $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}A[x\backslash a]))$-maximal knowledge stage $w^*$ which is $\mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\forall x.A))$-maximal because of corollary 1, we cannot conclude $w^* \Vdash A[x\backslash a]$ since $A[x\backslash a] \notin \mathcal{S}(\mathcal{G}(P\colon\ S, \mathbf{F}\forall x.A))$. The formula $A[x\backslash a]$ is, however, necessary in the premise of every $\mathbf{F}$-$\forall$ rule.

**The rules T-⇒ and T-¬:** These rules differ from the corresponding $\mathsf{TJ}$-rules since they introduce a $|\mathbf{T}$-formula. Semantically the conclusion $P\colon\ S, \mathbf{T}A \Rightarrow B$ of $\mathbf{T}$-$\Rightarrow$ represents a stage where a construction for every $\mathbf{T}$- and $|\mathbf{T}$-formula in $S$ is known while no construction is known for any of the $\mathbf{F}$-formulas. Additionally it is known how to construct evidence for $B$ whenever there is evidence for $A$.

$\mathbf{T}$-$\Rightarrow$ infers this situation from the premise that either evidence for $B$ is presently known or there is currently no evidence for $A$ but every *future* extension providing a construction for $A$ will also provide one for $B$. The corresponding $\mathsf{TJ}$-rule cannot make this distinction between current and future knowledge and therefore requires that $B$ can be constructed from $A$ (at the *current* stage) even if a construction for $A$ is assumed to be unknown. Our calculus, however, uses the $|\mathbf{T}$-sign in order to "hide" this requirement until we reach a stage where we may need it again, i.e. until a new construction is added to our knowledge. Technically this is performed by using the construct $S_{\mathbf{T}}$ in the premises of the rules $\mathbf{F}$-$\Rightarrow$-2, $\mathbf{F}$-$\neg$-2 and $\mathbf{F}$-$\forall$ which converts all $|\mathbf{T}$-formulas back into $\mathbf{T}$-formulas.

The main benefit of refining **T**-$\Rightarrow$ (and similarly **T**-$\neg$) is that its applicability (to the same formula) is now limited by the number of possible knowledge extensions. The latter is again limited by the number of different constructions that are introduced within a proof since $\mathsf{TJ}_A$-rules do not introduce constructions twice. In a tableau for a propositional formula $A$ this number must be finite since $A$ has only finitely many subformulas. Thus there can only be finitely many applications of **T**-$\Rightarrow$ and **T**-$\neg$ and the $\mathsf{TJ}_A$-search space must be finite (see sec. 4.2).

## 4 Properties of the system

In this section we present the most important properties of $\mathsf{TJ}_A$. Besides the fundamental properties of soundness and completeness w.r.t. intuitionistic validity we will briefly discuss search space complexity and decidability. Moreover we will use our system in order to determine formula classes for which classical and intuitionistic validity become equivalent. At some occasions we will refer to the *polarity* of a formula.

**Definition** **7** *Polarity*

Let $A$ be formula. The *polarity* $\mathtt{pol}(A')$ of any subformula $A'$ of $A$ is defined inductively as follows:

1. If $A' = A$ and $A$ is unsigned or $A$ is signed with **F** then $\mathtt{pol}(A') = 0$ otherwise $\mathtt{pol}(A') = 1$.
2. If $\mathtt{pol}(A')$ is defined and $A'$ is not atomic then the polarities of the immediate subformula(s) $A'_1$ as well as possibly $A'_2$ of $A'$ are defined as follows:
   (a) if the principal operator of $A'$ is binary then $\mathtt{pol}(A'_2) = \mathtt{pol}(A')$.
   (b) if $A' = \neg A'_1$ or $A' = A'_1 \Rightarrow A'_2$ then $\mathtt{pol}(A'_1) = 1 - \mathtt{pol}(A')$ otherwise $\mathtt{pol}(A'_1) = \mathtt{pol}(A')$. □

Informally the polarity of a subformula determines how it will be signed during a tableau construction. If the polarity is 0 then the subformula will occur **F**-signed and otherwise **T**-signed or **ıT**-signed.

### 4.1 Soundness and completeness

In [8] we have shown that our refinement of $\mathsf{TJ}$ is sound and complete.

**Theorem** **2** *Soundness of the system*

Let $A$ be a formula such that there is a $\mathsf{TJ}_A$-proof for $A$. Then $\models_j A$. □

**Theorem** **3** *Completeness of the system*

Let $A$ be a formula such that $\models_j A$. Then there is a $\mathsf{TJ}_A$-proof for $A$. □

The proofs for both theorems basically follow the line of the respective proofs for the soundness and completeness of $\mathsf{TJ}$ given in [3]. The informal justifications for the modifications of the $\mathsf{TJ}$-rules introduced by $\mathsf{TJ}_A$which we presented in the previous section were used at the respective places in a formalized manner. Since the proofs cover several pages we refer to our technical report [8] for details.

### 4.2 Deciding validity

According to the reflections at the end of section 3.2 the following theorem shows that intuitionistic validity can be decided by $\mathsf{TJ}_A$ for the propositional fragment:

**Theorem** **4** *Finite search space for propositional formulas*
Let $A$ be a formula with no quantifier. Then only finitely many $\mathsf{TJ}_A$-tableaux for $A$ can be constructed all of which being finite.

**Proof (sketch):** We introduce a well-founded ordering $<_p$ on PSSFs and show that after applying a propositional $\mathsf{TJ}_A$-rule to a given PSSF $P\colon\ S$ the result is smaller w.r.t. $<_p$. For this purpose we define two ordering functions $\mathrm{f}_A$ and $\mathrm{g}_A$ w.r.t. a given propositional formula $A$ which map $P\colon\ S$ as follows:

$$\mathrm{f}_A(P\colon\ S) = s(A) - l(P) \qquad\qquad \mathrm{g}_A(P\colon\ S) = \sum_{A' \in S} \deg(A')$$

Thereby $s(A)$ is the number of implicative and negated subformulas of $A$ having polarity 0, $l(P)$ is the length of $P$, $\deg(A')$ is the degree of $A'$ (i. e. the number of connectives in $A'$), and $A' \in S$ denotes that either $\mathbf{T}A' \in S$ or $\mathbf{F}A' \in S$.

Given two PSSFs $P_1\colon\ S_1$, $P_2\colon\ S_2$ within a tableau construction for $A$ we then say $P_1\colon\ S_1 <_p P_2\colon\ S_2$ iff either $\mathrm{f}_A(P_1\colon\ S_1) < \mathrm{f}_A(P_2\colon\ S_2)$ or $\mathrm{f}_A(P_1\colon\ S_1) = \mathrm{f}_A(P_2\colon\ S_2)$ but $\mathrm{g}_A(P_1\colon\ S_1) < \mathrm{g}_A(P_2\colon\ S_2)$

For a given PSSF $P\colon\ S$ within a tableau construction for a formula $A$ we obviously have $\mathrm{f}_A(P\colon\ S) \geq 0$. By a complete case analysis we can show that for any propositional rule the premises are always smaller w.r.t. $<_p$ than the conclusion (see [8] for details). Since no reduction rule can be applied anymore if $\mathrm{g}_A(P\colon\ S) = 0$, the tableau construction must terminate after finitely many steps. A fixed upper bound for the size of any possible tableau for $:\mathbf{F}A$ is given by $2^{n^2+n}$ where $n$ is the number of subformulas of $A$. After at most $2^n$ steps (reducing each subformula can yield at most two premises) a new construction must be added to the prefix if possible which can in turn happen at most $n$ times. At this point $\mathrm{f}_A$ will not become any smaller for any PSSF in the resulting configuration and we can perform at most another $2^n$ rule applications to each of these PSSFs until $\mathrm{g}_A = 0$. It is obvious that there can only be finitely many tableaux for $:\mathbf{F}A$ of a fixed maximum size. □

The above result can be extended to certain subclasses of predicate logic as well. Since we assume all formulas to be ground, semantical completeness is preserved if for any application of $\mathbf{F}$-$\exists$ or $\mathbf{T}$-$\forall$ to a PSSF $P\colon\ S$ the introduced parameter $c$ is selected only from $\mathcal{G}(P\colon\ S) \cup \{c_0\}$ where $c_0$ is an arbitrary parameter which is fixed throughout the proof construction. In this case $\mathbf{F}$-$\exists$ or $\mathbf{T}$-$\forall$ can be applied to $P\colon\ S$ only finitely many times since $\mathcal{G}(P\colon\ S)$ is finite. If in addition we can make sure that in any possible tableau construction we will reach a configuration where neither $\mathbf{F}$-$\forall$ nor $\mathbf{T}$-$\exists$ can be applied anymore from there on then the $\mathsf{TJ}_A$-search space starting from this configuration will be finite.

A class of formulas where any $\mathsf{TJ}_A$-tableau construction must lead to such a configuration can be described by the following syntactical criterion: any polarity 1 subformula of the form $\forall x.A_1$, $A_1 \Rightarrow A_2$, and $\neg A_1$ must not contain a polarity 1 subformula $\exists x.B_1$ or a polarity 0 subformula $\forall x.B_1$ and the same property must hold for every polarity 0 subformula of the form $\exists x.A_1$.

### 4.3 A connection with classical logic

In this section we will show an important correlation between classical and intuitionistic validity which follows from the specific properties of $\mathsf{TJ}_A$ and generalize results from [3, chapter 4§8]. For this purpose we define for a PSSF $P\colon S$ the set $\mathrm{C}(S)$ as $\{\mathbf{T}A | \mathbf{T}A \in S\} \cup \{\mathbf{F}A | \mathbf{F}A \in S\}$.

**Lemma** **2** *Intuitionistic vs. classical provability*

Let $P\colon S$ be a PSSF where $P$ has the form $\neg X.P'$ and no formula $A$ with $\circ A \in S$ ($\circ \in \{\mathbf{T}, \mathsf{I}\mathbf{T}, \mathbf{F}\}$) contains a subformula $\forall x.A'$ in $A$ with polarity 0.

There is a closed $\mathsf{TJ}_A$-tableau for $P\colon S$ iff there is a closed $\mathsf{TC}$-tableau for $\mathrm{C}(S)$.

**Proof (sketch):** Under the given assumptions $\mathbf{F}$-$\forall$, $\mathbf{F}$-$\Rightarrow$-0, $\mathbf{F}$-$\Rightarrow$-2, and $\mathbf{F}$-$\neg$-2 can never be applied during a $\mathsf{TJ}_A$-tableau construction for $P\colon S$. The behavior of the remaining $\mathsf{TJ}_A$-rules is virtually identical to the behavior of the respective $\mathsf{TC}$-rules which can be shown by a straightforward case analysis over the application of the remaining rules. We can thus define a one-to-one mapping between the respective rules of both systems and translate every rule application within one system into a corresponding one within the other. Hence, if a tableau for one system can be closed then so can a tableau for the other. □

This lemma provides a proof-theoretical foundation for the following result:

**Theorem** **5** *Intuitionistic vs. classical validity*

Let $A$ be a formula such that there is no subformula $\forall x.A'$ in $A$ with $\mathtt{pol}(\forall x.A') = 0$. Then $\models_j \neg A$ iff $\models_c \neg A$.

**Proof:** Any $\mathsf{TJ}_A$-proof for $\neg A$ will have to begin with $\mathbf{F}$-$\neg$-2 applied to $\{\colon \mathbf{F}\neg A\}$ yielding $\{\neg A\colon\ \mathbf{T}A\}$. By lemma 2 we know that there is a closed $\mathsf{TJ}_A$-tableau for $\neg A\colon\ \mathbf{T}A$ if and only if there is a closed $\mathsf{TC}$-tableau for $\mathrm{C}(\mathbf{T}A) = \mathbf{T}A$. This in turn is the case iff there is a closed $\mathsf{TC}$-tableau for $\mathbf{F}\neg A$ and hence a $\mathsf{TC}$-proof for $\neg A$. The theorem then follows from the soundness and completeness of $\mathsf{TJ}_A$ and $\mathsf{TC}$ w.r.t. intuitionistic and classical validity respectively. □

**Corollary** **2** *Constructibility vs. classical validity*

Let $A$ be a formula such that there is no subformula $\forall x.A'$ in $A$ with $\mathtt{pol}(\forall x.A') = 0$. Then $\models_j \neg\neg A$ iff $\models_c A$. □

Our results include formulas with universally quantified subformulas of polarity 1 and hence generalize theorem 8.3 in [3]. Furthermore we could achieve them on a proof-theoretical basis as opposed to the model-theoretical argumentation in [3].

### 4.4 Search space

In the proof for lemma 2 we have uncovered a general equivalence of proof search in $\mathsf{TJ}_A$ and $\mathsf{TC}$ for a special class of PSSFs, namely for those PSSFs $P: S$ where $P$ has the form $\neg X.P'$ and no formula $A$ with $\circ A \in S$ ($\circ \in \{\mathbf{T}, \mathbf{|T}, \mathbf{F}\}$) contains a subformula $\forall x.A'$ with polarity 0. This equivalence includes properties like confluence, invertibility and permutability of rule applications. Furthermore, tracing knowledge accumulation explicitly by handling prefixes contributes in general to a reduced applicability of the critical rules $\mathbf{T}$-$\Rightarrow$, $\mathbf{T}$-$\neg$, and the non-invertible rules $\mathbf{F}$-$\Rightarrow$-2 and $\mathbf{F}$-$\neg$-2.

A general statement about average search space reduction $\mathsf{TJ}_A$ achieves compared to $\mathsf{TJ}$ is difficult to obtain because it heavily depends on the shape of the formula to be proved. There are obviously classes of formulas where no search space reduction can be achieved at all. There are, however, also classes where the search space can be reduced exponentially. Consider, for instance, the following propositional formula for $n \geq 0$:

$$\neg \left[ \neg A_0 \wedge ((B_n \Rightarrow B_0) \Rightarrow A_n) \wedge \left( \bigwedge_{i=1}^{n} ((B_{n-1} \Rightarrow A_i) \Rightarrow A_{i-1}) \right) \right]$$

The $\mathsf{TJ}$-search space for this formula is infinite while the $\mathsf{TJ}_A$-search space is finite according to theorem 4. But even compared to the search space induced by the contraction-free intuitionistic proof systems presented in [2, 9] we achieve a $\mathcal{O}(n!)$-reduction. The reason for this is that a derivation in contraction-free calculi must backtrack over non-invertible reduction rules while in $\mathsf{TJ}_A$ this formula can be proved essentially by classical reasoning.

As a matter of fact, backtracking is *not* necessary for intuitionistic proof search on the given formula. It only becomes necessary *because* of the contraction-free rules. These rules can cause irreversible losses of assumptions which could be regained if contraction were supported. The purpose of the contraction-free rules is to provide finite search spaces for propositional formulas. $\mathsf{TJ}_A$ achieves the same property without the disadvantage of additional backtracking.

## 5 Conclusion, related and further work

In this paper we have presented a sound and complete proof system $\mathsf{TJ}_A$ for intuitionistic logic which supports classical reasoning in situations where constructive reasoning is unnecessary. We have shown that the $\mathsf{TJ}_A$-search space is finite for the propositional fragment and explained why this will also be the case for many other decidable formula classes. Moreover, we have demonstrated that our approach includes a potential for significant search space reductions over other intuitionistic proof systems. In addition to the computational properties of $\mathsf{TJ}_A$ we have used our calculus as a concept providing an extended perspective for theoretical insights into properties of intuitionistic logic e.g. the equivalence of classical and intuitionistic validity for a larger formula class than the one presented by Fitting [3].

Several approaches attempt to deal with the problem of avoiding to introduce the same construction several times. The contraction-free calculi for propositional (cf. [2, 6]) and for first-order intuitionistic logic (cf. [9]) have achieved considerable improvements to this problem. In particular, Hudelmaier's $\mathcal{O}(n \log n)$-space decision procedure [6] even avoids most of the additional backtracking problems we have mentioned in section 4.4. For several formula classes the "*certain falsehood*" concept presented in [9] yields smaller search spaces than $\mathsf{TJ}_A$. The major disadvantage of all contraction-free systems, however, is the use of transformation rules which violate the *subformula property* (cf. [4]) and are rather unnatural. Hence, there is no obvious way to adopt these approaches into matrix methods as presented in [17, 12] or translation methods as presented in [11, 10]. In contrast to that, our approach preserves the subformula property and provides a more natural foundation for the development of efficient proof methods.

The "*ft*" theorem prover [13] introduces the concept of *transfer lists* together with *implication locking* which have similarities to our prefix handling and **T**-formula concepts respectively. However, they are used there as mechanisms to control proof search in a $\mathsf{TJ}$-like calculus rather than as a part of the proof theory as in $\mathsf{TJ}_A$. Moreover, ft does not incorporate any technique to exploit maximality properties for explicit use of classical proof search at suitable situations.

Our perspective for further work includes the development of matrix and translation methods based on $\mathsf{TJ}_A$ which is currently in progress. A first spin-off from these efforts is presented in [7]. Additionally we are working on several improvements to the $\mathsf{TJ}_A$-system. As mentioned in section 4.2 a suitable concept of reducing the number of possible $\mathbf{F}$-$\exists$ and $\mathbf{T}$-$\forall$ applications is desirable in order to obtain decidability for a considerably larger class of formulas. Other improvements could include the integration of the above-mentioned concept of certain falsehood or refinements of the $A$"$\in$"$P$-constraints for the $\mathbf{F}$-$\Rightarrow$-$i$ and $\mathbf{F}$-$\neg$-$i$ rules. We strongly believe that the $\mathsf{TJ}_A$-system provides a suitable framework for comprehensive investigations on the actual difference between constructive and classical reasoning which could lead to considerable improvements of automated proof search in intuitionistic logic.

## References


1. W. Bibel: *Automated Theorem Proving,* Braunschweig, $^2$1987
2. R. Dyckhoff: *Contraction-Free Sequent Calculi for Intuitionistic Logic,* Journal of Symbolic Logic, Vol. 57, No. 3, pp. 795–807, 1992
3. M. Fitting: *Intuitionistic Logic Model Theory and Forcing,* Amsterdam, $^1$1969
4. G. Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935.
5. A. Heuerding, e. a.: *Propositional Logics on the Computer,* Proceedings of the 4th TABLEAUX '95, LNAI 918, pp. 310–323-, 1995
6. J. Hudelmaier: *An $O(n \log n)$-space decision procedure for intuitionistic propositional logic,* In: *Journal of Logic and Computation* 3(1)1, pp. 63–75, 1993

7. D. Korn, C. Kreitz.: *Efficiently Deciding Intuitionistic Propositional Logic via Translation into Classical Logic.,* Tech. Rep. AIDA-96-10, TH Darmstadt, 1996.
8. D. Korn, C. Kreitz.: *A Constructively Adequate Refutation System for Intuitionistic Logic,* Tech. Rep. AIDA-96-14, TH Darmstadt, 1996.
9. P. Miglioli, U. Moscato, M. Ornaghi: *An Improved Refutation System for Intuitionistic Predicate Logic,* Journal of Automated Reasoning 13, pp 361–373, 1994
10. A. Nonnengart: *Resolution-Based Calculi for Modal and Temporal Logics,* CADE-13, LNAI 1104, pp. 598–612, Springer, 1996
11. H. J. Ohlbach: *Semantics–Based Translation Methods for Modal Logics,* Journal of Logic and Computation, Vol. 1, no. 6, pp 691–746, 1991
12. J. Otten, C. Kreitz: *A Connection Based Proof Method for Intuitionistic Logic,* Proceedings of the 4th TABLEAUX '95, LNAI 918, pp. 122–137, 1995
13. D. Sahlin, T. Franzén, S. Haridi: *An Intuitionistic Predicate Logic Theorem Prover,* Journal of Logic and Computation, Vol. 2, no. 5, pp 619–656, 1992
14. K. Schütte: *Vollständige Systeme modaler und intuitionistischer Logik,* Springer–Verlag, Berlin, 1968
15. R. Smullyan: *First-Order Logic,* Springer, $^2$1968
16. T. Tammet: *A Resolution Theorem Prover for Intuitionistic Logic,* CADE-13, LNAI 1104, pp. 2–16, Springer, 1996
17. L. Wallen: *Automated Proof Search in Non–Classical Logics,* The MIT Press, $^1$1990