A Unified Approach to Constructive and Recursive Analysis

by

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1. Introduction

Many mathematicians familiar with the constructivistic objections
to classical mathematics concede their validity but remain
unconvinced that there is any satisfactory alternative.

Among others Bishop [2] and Bridges [3] showed that large parts
of classical analysis and functional analysis can be formulated
in a constructive way. But it does not seem to be likely that
their concept of constructivity will be generally accepted.

The previous attempts to formulate and study effectivity in
analysis can roughly be divided into three classes. The con-
structivists only study "constructive" objects and only use
"constructive" proofs e.g. by using intuitionistic logic
(Brouwer [4], Lorenzen [15], Bishop [2], Bridges [3], et al.).
The other two attempts are based on recursion theory. The
"Russian school" (Celtin [5], Kushner [14], Aberth [1], et al.)
starts with an "effective" partial numbering of the set of
computable real numbers \( \mathbb{R}_c \) by which computability on \( \mathbb{N} \)
is transferred to \( \mathbb{R}_c \). The "Polish school" (Grzegorczyk [8],
Klaus [9], et al.) starts (essentially) with an "effective"
representation of all real numbers by \( \mathbb{I} := \mathbb{N}^\mathbb{N} \), by which
computability of operators on \( \mathbb{I} \) is transferred to compu-
tability on \( \mathbb{R} \). The approach presented here is a consequent
continuation of that of the Polish school. It is formulated
as a theory of numberings \( \nu: \mathbb{N} \rightarrow S \) (Ershov [7]) and of
representations \( \delta: \mathbb{I} \rightarrow M \) (Kreitz and Weihrauch [12]) and
admits to study continuity, computability and computational
complexity (Ko [10]) which can be considered as different
degrees of constructivity. We will only outline basic
definitions and properties and show by examples how analysis
can be developed in this context.
We shall consider the set $\mathbb{N}$ of natural numbers as the (w.l.g. single) concrete set of finite objects and the set $\mathbb{F} = \mathbb{N}^\mathbb{N}$ of sequences of natural numbers as the (w.l.g. single) concrete set of infinite objects on which "constructions" can be performed (e.g. by a computer or by men with paper and pencil). Constructions on $\mathbb{F}$ are governed by Baire's topology, thus continuity of functions $\mathbb{F} \longrightarrow \mathbb{F}$ or $\mathbb{F} \longrightarrow \mathbb{N}$ (in the discrete topology) is the weakest form of constructivity. Computability is the next additional stronger requirement for functions $\mathbb{F} \longrightarrow \mathbb{F}$, $\mathbb{F} \longrightarrow \mathbb{N}$, or $\mathbb{N} \longrightarrow \mathbb{N}$. Further conditions on the computational complexity (e.g. primitive recursive or polynomial) yield more restricted kinds of constructivity. As a basis for the studies in Chapter 2 a unified concise Type 2 theory of continuity and computability on $\mathbb{F}$ is outlined, which is formally similar to ordinary Type 1 recursion theory on $\mathbb{N}$. More details can be found in a forthcoming paper (Weihrauch [23]).

For all other objects the elements of $\mathbb{N}$ or $\mathbb{F}$ are used as names. Let $S$ be a set to be named by numbers. Then any $s \in S$ must have a name and any number is name of at most one $s \in S$. Thus a naming of a set $S$ by numbers is a possibly partial surjective function $v: \mathbb{N} \longrightarrow S$, which we call a numbering. The theory of numberings is studied in detail by Ershov [7] (see also Mal'cev [16]). Similarly a naming of a set $M$ by elements of $\mathbb{F}$ is a possibly partial surjective function $\delta: \mathbb{F} \longrightarrow M$ which we call a representation. Constructivity on $S$ or $M$ is defined via constructivity on concrete objects, namely the names w.r.t. given numberings or representations.

Chapter 3 gives an outline of a general theory of representations. An essential point is the definition of admissible representations of separable $T_0$-spaces. Again in this theory topological (t-) and computational (c-) aspects are considered simultaneously (Kreitz and Weihrauch [12]).
In Chapter 4 as an example representations of the real numbers are studied. It is shown that the significant differences between previously defined representations are topological ones. As a further application compactness on \( \mathbb{R} \) will be studied in Chapter 5.

2. Type 2 Recursion Theory

As we already have outlined computability and continuity on \( \mathbb{N} \) and \( \mathbb{F} \) are the basis of our approach to constructive and recursive analysis. We assume the reader is familiar with ordinary recursion theory on \( \mathbb{N} \) and with basic properties of numberings (Mal'cev [16], Rogers [20], Ershov [7]). Let \( \varphi \) be a standard numbering of the unary partial recursive functions \( P^{(1)} \), let \( < > : \mathbb{N}^n \rightarrow \mathbb{N} \) be Cantor's \( n \)-tupling function. By \( f: A \langle \ldots \rangle B \) (with dotted arrow) we denote a possibly partial function from \( A \) to \( B \). Unlike to ordinary (Type 1) recursion theory for Type 2 recursion theory there is no generally accepted formalism. Below we outline a unified approach which is formally similar to the Type 1 formalism. More details can be found in Weihrauch's paper [23].

We start with some topological preliminaries. Let \( \mathbb{F} := \mathbb{N}^\mathbb{N} \), \( \mathbb{B} := \mathbb{F} \cup W(\mathbb{N}) \) where \( W(\mathbb{N}) \) is the set of words (i.e. finite sequences) over \( \mathbb{N} \). On \( \mathbb{B} \) a partial order is defined by \( b \preceq c :<=> b \) is a prefix of \( c \). On \( \mathbb{B} \) we shall assume the topology corresponding to the cpo \( (\mathbb{B}, \preceq, \xi) \) (Egli and Constable [6], Scott [21]). The topology induced on the subset \( \mathbb{F} \) is Baire's topology. On \( \mathbb{N} \) we consider the discrete topology.

First a standard representation \( \psi \) of \( [\mathbb{F} \rightarrow \mathbb{B}] \), the set of continuous functions from \( \mathbb{F} \) to \( \mathbb{B} \) is defined. From \( \psi \) we derive representations of certain continuous functions from \( \mathbb{F} \) to \( \mathbb{F} \) and from \( \mathbb{F} \) to \( \mathbb{N} \). The construction of \( \psi \) rests on the following property. Let \( \gamma := W(\mathbb{N}) \rightarrow W(\mathbb{N}) \) be isotone (w.r.t. \( \xi \)). Then the function \( \tilde{\gamma}: \mathbb{F} \rightarrow \mathbb{B} \), defined by
\(\overline{\gamma}(p) := \sup(\gamma(w) | w \in p)\), is continuous. And for any continuous function \(\Gamma : \mathcal{F} \rightarrow \mathcal{B}\), \(\Gamma = \overline{\gamma}\) for some isotone \(\gamma : W(\mathbb{N}) \rightarrow W(\mathbb{N})\).

The function \(\gamma\) specifies, how from prefixes of \(p \in \mathcal{F}\) sufficiently many prefixes of \(f(p) = \overline{\gamma}(p)\) can be determined.

A function \(\Gamma : \mathcal{F} \rightarrow \mathcal{B}\) is called computable, iff \(\Gamma = \overline{\gamma}\) for some computable function \(\gamma\). The computable functions \(\Gamma : \mathcal{F} \rightarrow \mathcal{B}\) can easily be characterized by oracle Turing machines which on input \(p \in \mathcal{F}\) from time to time read a value \(p(i)\) and from time to time write one of the values \(q(0), q(1), \ldots\) (in this order) of the result \(q \in \mathcal{B}\). For transforming \(n\)-ary functions on \(\mathcal{F}\) to unary ones, the following tupling functions \(\Pi^{(n)} : \mathcal{F} \rightarrow \mathcal{F}\) are used:

\[
\Pi(p, q)(i) := \begin{cases} 
   p(x) & \text{if } i = 2x, q(x) \text{ if } i = 2x+1
\end{cases}, \Pi^{(1)}(p) := p
\]

\[
\Pi^{(n+1)}(p_1, \ldots, p_{n+1}) := \Pi(\Pi^{(n)}(p_1, \ldots, p_n), p_{n+1})
\]

notation: \(<p_1, \ldots, p_n> := \Pi^{(n)}(p_1, \ldots, p_n)\).

Also \(\omega\)-ary tupling is possible: \(\Pi^{(\omega)}(p_0, p_1, \ldots, <i, j> := p_i(j)\).

The functions \(\Pi^{(n)}\) and \(\Pi^{(\omega)}\) are homeomorphisms w.r.t. the product topologies. The projections of their inverses are computable.

The definition of \(\overline{\gamma}\) is effective in the following sense.

There is a computable (by an oracle Turing machine) operator \(\Gamma_u : \mathcal{F} \rightarrow \mathcal{B}\) with the following property. On input \(p, q\) it determines \(\overline{\gamma}(q)\) if \(\gamma := v_N p v_N^{-1}\) (\(v_N\) is a bijective standard numbering of \(W(\mathbb{N})\)) is isotone, \(\Gamma(q)\) for some continuous \(\Gamma : \mathcal{F} \rightarrow \mathcal{B}\) otherwise. Then by \(\psi(p)(q) := \psi(p)(q) = \Gamma_u \langle p, q \rangle\) a representation \(\psi : \mathcal{F} \rightarrow [\mathcal{F} \rightarrow \mathcal{B}]\) of the continuous functions from \(\mathcal{F}\) to \(\mathcal{B}\) is defined, which satisfies the "universal Turing machine theorem" and the "smn-theorem".
Theorem:

1. $\psi_p(q) = \Gamma_u(p, q)$ for some computable $\Gamma_u \in [\mathbb{F} \to \mathbb{N}]$
2. $\psi\langle q, r \rangle = \psi_{\langle p, q \rangle}(r)$ for some computable $\Sigma \in [\mathbb{F} \to \mathbb{N}]$ with range $(\Sigma)q \subseteq \mathbb{F}$.

The proof is similar to that in ordinary recursion theory.

Notice that $\Gamma_u$ and $\Sigma$ are not only continuous but even computable. Similar to Type 1 recursion theory the utm-theorem and the smn-theorem characterize the representation $\psi$ uniquely up to (computable) equivalence (see Chapter 3). More interesting than $\psi$ itself are two representations derived from $\psi$.

Definition:

1. Define a set $[\mathbb{F} \to \mathbb{N}]$ of partial functions from $\mathbb{F}$ to $\mathbb{N}$ and a representation $\chi: \mathbb{F} \to [\mathbb{F} \to \mathbb{N}]$ by:
   
   $\chi_p(q) := \chi(p)(q) := \text{(div if } \psi_p(q) = e \in \mathbb{N}, \text{ the first number of the sequence } \psi_p(q) \text{ otherwise).}$

2. Define a set $[\mathbb{F} \to \mathbb{F}]$ of partial functions from $\mathbb{F}$ to $\mathbb{F}$ and a representation $\tilde{\psi}: \mathbb{F} \to [\mathbb{F} \to \mathbb{F}]$ by:

   $\tilde{\psi}_p(q) := \tilde{\psi}(p)(q) := (\psi_p(q) \text{ if } \psi_p(q) \in \mathbb{F}, \text{ div otherwise).}$

This definition extends well known concepts of computable operators and functionals to a uniform topological description, where the elements computable w.r.t. a given representation are those with computable names. The functions from $[\mathbb{F} \to \mathbb{N}]$ and from $[\mathbb{F} \to \mathbb{F}]$ have natural domains (c.f. domains of partial recursive functions). But the set of domains is sufficiently rich such that any continuous function is essentially considered.

Theorem:

1. $[\mathbb{F} \to \mathbb{N}]$ is the set of all continuous functions $\Sigma: \mathbb{F} \to \mathbb{N}$ such that $\text{dom}(\Sigma)$ is open. For any continuous function $\Gamma: \mathbb{F} \to \mathbb{N}$ there is some $\Sigma \in [\mathbb{F} \to \mathbb{N}]$ which extends $\Gamma$.
2. A valid statement is obtained by substituting "\mathbb{N}" by "\mathbb{F}" and "open" by "$G_\delta$-subset" in (1).
Also the representations $\chi$ and $\tilde{\psi}$ satisfy the utm- and the smn-theorem. This leads to a rich theory of continuity and of computability which is formally similar to Type 1 recursion theory. From the above theorem we conclude that by $\omega'(p) := \text{dom}(\chi_p)$ a representation $\omega'$ of the open subsets of $\mathcal{F}$ is defined, which corresponds to the numbering $i \mapsto \text{dom}(\phi_i)$ of the r.e. subsets of $\mathbb{N}$. We call a subset $A \subseteq \mathcal{F}$ t-open (c-open) iff $A = \omega'(p)$ for some (computable) $p$. $A$ is t-clopen (c-clopen), iff $A$ and $\mathcal{F}\setminus A$ are t-open (c-open). The t-open (c-open) sets are exactly the projections of the t-clopen (c-clopen) sets.

The self-applicability and the halting problem of $\chi$ can be formulated. They are c-open, not t-clopen, c-complete and c-productive. Also effective inseparability can be defined. The sets $\{p|\chi_p(p) = 0\}$ and $\{p|\chi_p(p) = 1\}$ are c-effectively inseparable. This property can be used in the study of precomplete representations. Many other properties can be proved easily but more questions are still unsolved in this theory of continuity and computability on $\mathcal{F}$.

3. Theory of representations

In order to define computability and constructivity on a set $M$ with cardinality not greater than that of the continuum, we represent $M$ by a surjective mapping $\delta: \mathcal{F} \rightarrow M$, called representation of $M$. Some examples for representations are the enumeration representation $\mathbb{M}: \mathcal{F} \rightarrow P^\omega_w$ with $\mathbb{M}(p) := \{i|i+1 \in \text{range } p\}$, the representation $\delta_{cf}$ of $P^\omega_w$ by characteristic functions with $\delta_{cf}(p) := \{i|p(i) = 0\}$, and the representations $\psi: \mathcal{F} \rightarrow [\mathcal{F} \rightarrow \mathbb{B}]$, $\tilde{\psi}: \mathcal{F} \rightarrow [\mathcal{F} \rightarrow \mathcal{F}]$, $\chi: \mathcal{F} \rightarrow \mathcal{F}$, $\omega': \mathcal{F} \rightarrow \mathbb{N}$, $\mu: \mathcal{F} \rightarrow \{x \in \mathcal{F}|x \text{ is open}\}$ introduced in chapter 2.

Effectivity properties of theorems, functions, sets, predicates etc. can be expressed by effectivity of correspondences (i.e. multivalued functions) which are triples $f = (M,M',P)$ where $P \subseteq M \times M'$.
Definition:
Let $\delta, \delta'$ be representations of $\mathcal{M}$ resp. $\mathcal{M}'$ and let $f = (M, M', P)$ be a correspondence. $f$ is called weakly $(\delta, \delta')$-t-(c-) effective iff there is some (computable) $\Gamma \in \mathcal{IF} \rightarrow \mathcal{IF}$ such that

$$(\delta q, \delta' q) \in P \text{ for all } q \in \delta^{-1} \text{dom}(f).$$

$f$ is called $(\delta, \delta')$-t-(c-) effective, iff in addition $\Gamma(q)$ is undefined for all $q \in \delta^{-1}(M \setminus \text{dom } f)$.

$(\delta, \nu)$-effectivity of a correspondence $f = (M, S, P)$ where $\nu$ is a numbering of $S$ is defined accordingly using $[\mathcal{IF} \rightarrow \mathcal{N}]$ instead of $[\mathcal{IF} \rightarrow \mathcal{IF}]$. For convenience we shall say "continuous" instead of "t-effective" and "computable" instead of "c-effective".

Since a partial function is a single valued correspondence the above definition is applicable to functions. A subset $A \subseteq \mathcal{M}$ can either be characterized as the domain of a partial function or by its characteristic function.

A set $A \subseteq \mathcal{M}$ is called $\delta$-(c-) open iff $d_A := (M, \mathcal{N}, A \times \mathcal{N})$ is $(\delta, \text{id}_\mathcal{N})$-t-(c-) effective. $A$ is called $\delta$-t-(c-) clopen iff $c_A := M, \mathcal{N}, \{(x, 0) \mid x \in A\} \cup \{(y, 1) \mid y \in M \setminus A\}$ is $(\delta, \text{id}_\mathcal{N})$-t-(c-) effective.

Usually we say "provable" instead of "c-open" and "decidable" instead of "c-open".

The $\delta$-effectivity on $\mathcal{M}$ strongly depends on the representation $\delta$. Consider the two questions whether complementation on $\mathcal{P}_\mathcal{M}$ is effective and whether countable union on $\mathcal{P}_\mathcal{M}$ is effective. There is no absolute answer but only one relative to the considered representation: Complementation is $(\delta_{cf}, \delta_{cf})$-computable but not even $(\mathcal{M}, \mathcal{M})$-continuous, countable union is computable w.r.t. $\mathcal{M}$ but not even weakly continuous w.r.t. $\delta_{cf}$ (use $\mathcal{N}(\omega)$ for formalization). This difference can be explained using the intuitive concept of finitely (or continuously) accessible (f.a.) information. Every true information $n \in \mathcal{M}(p)$ is f.a. from $p$, no true information $n \notin \mathcal{M}_p$ is f.a. from $p$. But every true
information $n \in \delta_{cf}(p)$ or $m \notin \delta_{cf}(p)$ is f.a. from $p$.
Representations may be changed in a certain way without changing the induced effectivity.

For any two representations $\delta, \delta'$ of $M$ resp. $M'$ define

$$\delta \leq_t \delta' : \iff M \subseteq M' \text{ and } id_{M,M'} \text{ is } (\delta, \delta') \text{-effective,}$$

$$\delta \equiv_t \delta' : \iff \delta \leq_t \delta' \text{ and } \delta' \leq_t \delta.$$

c-reducibility ($\leq_c$) and c-equivalence ($\equiv_c$) are defined accordingly.
It is easy to show that $\delta_{cf} \leq_c M$ and that $M$ and $\delta_{cf}$ are not t-equivalent.

Since effective functions are closed under composition two representations are t-(c-) equivalent if and only if they define the same continuity (computability theory).

Theorem:
Let $\delta, \delta'$ be representations of $M$. Then (1), (2) and (3) are equivalent.

(1) $\delta \leq_t \delta'$
(2) For any representation $\delta_1 : \mathcal{F} \to M_1$ and any correspondence $f = (M_1, M, p)$:
    $f$ (weakly) $(\delta_1, \delta)$ t-effective $\Rightarrow f$ (weakly) $(\delta_1, \delta')$ t-effective
(3) For any representation $\delta_2 : \mathcal{F} \to M_2$ and any correspondence $g = (M_2, M, p)$:
    $g$ (weakly) $(\delta, \delta_2)$ t-effective $\Rightarrow g$ (weakly) $(\delta, \delta')$ t-effective.

Proof:
(1) $\Rightarrow$ (2) $\Rightarrow$ (3): Immediately from "$\delta \leq_t \delta' \Rightarrow id_M$ is $(\delta, \delta')$-continuous"

(2) $\Rightarrow$ (1): Choose $\delta_1 := \delta$ and $f := id_M$
(3) $\Rightarrow$ (1): Choose $\delta_2 := \delta'$ and $g := id_M$

Every representation $\delta : \mathcal{F} \to M$ induces a topology $\tau_\delta$ on $M$ by $x \in \tau_\delta : \iff \delta^{-1}(x) = A \cap \text{dom } \delta$ for some open subset $A \subseteq \mathcal{F}$. 

\( \tau_\delta \) is called the final-topology of \( \delta \) and it consists exactly of all the \( \delta \)-open subsets of \( M \). For example \( \tau_M \), the final topology of the enumeration representation of \( P_\omega \) is determined by the basis \( \{ O_e \mid e \leq N, \text{finite} \} \) where \( O_e := \{ x \leq N \mid e \leq x \} \). Clearly \( \tau \)-equivalent representations have the same final topologies but the converse does not hold in general (a counterexample is presented in chapter 4). If on \( M \) already a topology \( \tau \) is given then \( \tau = \tau_\delta \) should hold for any "reasonable" representation of \( M \). (In some special cases there might be reasons for choosing \( \tau_\delta \neq \tau \).) For separable \( T_0 \)-spaces representations equivalent to a standard-representation defined as follows seem to be the most natural ones.

**Definition:**

Let \( (M, \tau) \) be a separable \( T_0 \)-space and let \( U \) be a numbering of some basis of \( \tau \). For \( x \in M \) let \( \varepsilon_u(x) := \{ i \mid x \in U_i \} \).

A standard-representation \( \delta_u \) of \( (M, \tau) \) is defined by

\[
\text{dom} \delta_u := M^{-1} \varepsilon_u(M) \quad \text{and} \quad \delta_u(p) := \varepsilon_u^{-1} M(p) \quad \text{whenever} \quad p \in \text{dom} \delta_u.
\]

A standard-representation \( \delta_u \) of a separable \( T_0 \)-space has some remarkable properties:

**Lemma:**

(1) \( \delta \) is continuous and open, especially \( \tau = \tau_u \).

(2) For any space \( (M', \tau') \) and any \( H : M \rightarrow M' \)

\( H \circ \delta_u \) is continuous \( \iff \) \( H \) is continuous.

(3) \( \zeta \leq \tau \delta_u \) for any continuous \( \zeta : \mathbb{I}^F \rightarrow M \).

**Proof:**

(1) \( \varepsilon_u \) is \( (\tau, \tau_M) \)-continuous and open and \( M \) is open and continuous w.r.t. \( \tau_M \).

(2) Follows from (1).

(3) Define \( \Delta : \mathbb{I}^F \rightarrow \mathbb{I}^F \) by

\[
\Delta(p) < n, k > := (n + d \text{ if } \zeta[p[k]] \in U_n, 0 \text{ otherwise})
\]

Then \( M^\Delta(p) = \{ n \mid \zeta(p) \in U_n \} = \varepsilon_u(\zeta(p)) \) for every \( p \in \text{dom} \zeta \).

Therefore \( \zeta \leq \tau \delta_u \) by \( \Delta \).
An immediate consequence ist that all the standard-representations of the same space are \( t \)-equivalent and therefore the equivalence class \( \{ \delta | \delta \equiv _t \delta \} \) does not depend on the numbering \( U \). Since every representation equivalent to \( \delta _u \) induces the same continuity-theory we call a representation \( \delta \) of a separable \( T_0 \)-space \( t \)-effective (or admissible) iff \( \delta \equiv _t \delta _u \) for some standard-representation \( \delta _u \). The representations \( M \) and \( \delta _{cf} \) of \( \mathcal{T}_\omega \) are admissible. The decimal-representation of the real numbers is not (see next chapter).

For admissible representations of a space \((M,τ)\) the final topology is identical with \( τ \). Furthermore topological continuity and continuity w.r.t. these representations are closely related.

**Theorem:**

Let \((M_1,τ_1)\) be separable \( T_0 \)-spaces and let \( δ_i : \mathbb{IF} → M_i \) be admissible representations \((i = 1,2)\). Let \( F : M_1 → M_2 \), then:

1. \( F(τ_1,τ_2) \)-continuous \( ↔ \) \( F \) weakly \( (δ_1,δ_2) \)-continuous,
2. \( F(τ_1,τ_2) \)-continuous \( ∧ \) \( \text{dom} F \in G_δ(τ_1) \rightarrow F(δ_1,δ_2) \)-continuous.

**Proof:**

W.l.g. we may assume \( δ_1 \) and \( δ_2 \) to be standard representations.

1. Let \( F : M_1 → M_2 \) be \( (τ_1,τ_2) \)-continuous and let \( δ' := F ∘ δ_1 \).
   
   Then \( δ' : F → M \) is continuous and therefore \( δ' ≤ _t δ_2 \).
   
   i.e. \( Fδ_1(p) = δ_2 Γ(p) \) for all \( p ∈ \text{dom} Fδ_1 \) with some continuous \( Γ \). Conversely let \( F \) be weakly \( (δ_1,δ_2) \)-continuous.
   
   i.e. \( Fδ_1 = δ_2 Γ \) for some continuous \( Γ \). Since \( δ_2 \) is continuous the same holds for \( Fδ_1 \) and hence also \( F \) is continuous.

2. Let \( F \) be continuous and \( \text{dom} F ∈ G_δ(τ_1) \). i.e. \( \text{dom} F = \bigcap _{i \in \mathbb{N}} O_i \)
   
   where \( O_i \in τ_1 \) for \( i \in \mathbb{N} \). Since \( δ_1 \) is continuous there are sets \( O'_i \) open in \( \mathbb{IF} \) such that \( \text{dom} Fδ_1 = δ_1^{-1} \text{dom} F = \bigcap _{i \in \mathbb{N}} O'_i ∩ \text{dom} δ_1 \).

By (1) there is some \( Γ ∈ [IF → IF] \) with \( Fδ_1(p) = δ_2 Γ(p) \) for all \( p ∈ \text{dom} Fδ_1 \). Now let \( Γ_1 \) be the restriction of \( Γ \) to the \( G_δ \)-set \( ∩ O_i \). Then \( \text{dom} Γ_1 = \text{dom} Γ ∩ O_i \) is a \( G_δ \)-set and hence \( Γ_1 ∈ [IF → IF] \) and for every \( p ∈ \text{dom} δ_1 \).
p \in \text{dom}\delta_1 \Rightarrow \delta_2 \Gamma_1 (p) = \delta_1 (p)
\text{and } p \notin \text{dom}\delta_1 \Rightarrow p \notin \text{dom}\Gamma_1.

This means \text{F} is strongly \((\delta_1, \delta_2)\)-continuous.

□

For some representations the converse of (2) also holds (e.g.,
for the representation \(\rho\) of \(\mathbb{R}\) by normed Cauchy-sequences—see
next chapter).

There are many other aspects of representations which should be
studied, for example recursion-theoretic properties, computable
elements, the structure of equivalence degrees, closure proper-
ties etc. There are also natural representations the final topo-
ologies of which are not separable. See Kreitz & Weihrauch [12]
for further discussion.

4. Representations of the Real Numbers

An excellent recursion theoretic comparison of many representa-
tions of \(\mathbb{R}\) has been given by T. Dell [25]. In this chapter
we show by examples that the essential differences between the
representations which are mainly discussed in constructive and
computable analysis already are of topological nature. More de-
tails can be found in a paper by the authors [24].

Let \(\mathbb{R}\) be the set of real numbers, let \(\tau_\mathbb{R} \subseteq 2^{\mathbb{IR}}\) be the set
of open subsets \(A \subseteq \mathbb{R}\), where \(A\) is open iff it is the union
of open intervals \((x,y) := \{z \in \mathbb{R} | x < z < y\}\). The space \((\mathbb{R}, \tau_\mathbb{R})\)
is a separable \(T_0\)-space. From Chapter 3 we know that there are
admissible representations \(\delta\) of \(\mathbb{R}\) for which especially
\(\tau_\delta = \tau_\mathbb{R}\). The authors have shown [12] that for any complete
separable metric space an admissible representation can be
defined via normed Cauchy sequences. In the case of \(\mathbb{R}\) a
useful representation of this type is as follows:
**Definition:**

Let \( Q_n := \{m \cdot 2^{-n} \mid m \in \mathbb{Z} \} \), \( Q_d := \cup Q_n \), let \( v_d \) be a standard numbering of \( Q_d \). Then the standard representation \( \rho \) of the real numbers is defined by \( \text{dom}(\rho) := \{ p \in \mathbb{R} \mid (\forall k)(v_d p(k) \in Q_k \land v_d p(k + 1) < 2^{-k}) \} \),

\[
\rho(p) := \lim v_d p(n) \quad \text{for all } p \in \text{dom}\rho.
\]

This representation is admissible and its final topology is \( \tau_R \).

As a product of admissible representations \( \rho^n \), defined by \( \rho^n(p_1, \ldots, p_n) := (\rho(p_1), \ldots, \rho(p_n)) \), is admissible for any \( n > 0 \). Therefore \( \rho^n \) satisfies Rice's theorem which states that \( \rho^{-1}X \) is not t-clopen if \( X \in \mathbb{R}^n \) is not trivial. Especially relations like \( x < y, x = y \ldots \) are not \( \rho^2 \)-decidable.

The definition of the real numbers as cuts of rational numbers induces several representations.

**Definition:**

Define a representation \( \rho_\prec \) of \( \mathbb{R} \) as follows:

\[
\text{dom} \rho_\prec := \{ p \in \mathbb{R} \mid \exists v_d m_p \text{ has an upper bound} \}
\]

\[
\rho_\prec(p) := \sup v_d m_p \quad \text{for } p \in \text{dom}(\rho_\prec).
\]

A representation \( \rho_\succ \) is defined correspondingly by \( \rho_\succ(p) = \inf v_d m_p \).

The representations \( \rho_\prec \) and \( \rho_\succ \) are admissible w.r.t. their final topologies \( \tau_\prec \) and \( \tau_\succ \), respectively, where

\[
\tau_\prec = \{(x; \infty) \mid x \in \mathbb{R} \cup \{-\infty\} \}, \quad \tau_\succ = \{(\infty; x) \mid x \in \mathbb{R} \cup \{\infty\} \}.
\]

It is easy to show that \( \rho \in \inf_c(\rho_\prec, \rho_\succ) \) (c.f. [24]) and by the characterization of the final topologies \( \rho_\prec^L p, \rho_\succ^L p, \rho_\prec^L p \) and \( \rho_\succ^L p_\prec \). The characteristic properties of \( \rho, \rho_\prec \) and \( \rho_\succ \) are given by the finitely accessible information in each case. For any \( i \in \mathbb{N} \) and \( p \in \mathbb{R} \) the property...
\( \nu_D(i) < \rho_c(p) \) can be proved in finitely many steps iff it is true, more precisely \( \{(d,x) \in Q_D \times IR \mid d < x\} \) is \([\nu_D, \rho_c]\)-provable. \( \rho_c \) is the greatest representation (w.r.t. \( \leq_c \) or \( \leq_t \)) with this property (similarly for \( \rho_c \)). Finally \( \rho \) is the greatest representation \( \delta \) of \( IR \) such that

\[
\{(d,e,x) \in Q_D \times Q_D \times IR \mid d < x < e\} \text{ is } [\nu_D, \nu_D, \delta]\text{-provable.}
\]

Classical analysis suggests the following representation

\( \delta_c \) by unrestricted Cauchy sequences: \( \text{dom}(\delta_c) = \{p \mid \nu_D p \text{ is a Cauchy sequence}\} \), \( \delta_c(p) := \lim_{n \to \infty} p(n) \) for \( p \in \text{dom}(\delta_c) \). But in this case no prefix \( p[n] \) of \( p \) gives any information of \( \delta_c(p) \), no information is finitely accessible. This is formally expressed by the characterization of the final topology \( \tau_c \) of \( \delta_c \): \( \tau_c = \{\emptyset, IR\} \), i.e. the indiscrete topology on \( IR \). Therefore, \( \delta_c \) is not useful in constructive analysis for purely topological reasons.

The most commonly used representations are the \( r \)-adic representations (\( r = 2, 8, 16, \ldots \)). The finite \( r \)-adic fractions are dense in \( IR \), the infinite \( r \)-adic fractions, however, are not appropriate for representing the real numbers. Again, this has topological reasons. For simplicity we consider the case \( r = 10 \).

Define \( \delta_{DEZ} \) by \( \text{dom}(\delta_{DEZ}) = \{p \in IF \mid (\forall n > 0)(p(n) < 10)\}, \)

\[
\delta_{DEZ}(p) := (\pi_1p(0) - \pi_2p(0)) + \sum_{i \in \mathbb{N}} \{p(i) \cdot 10^{-i} \mid i \in \mathbb{N}\}.
\]

It can be shown that \( \delta_{DEZ} \) has the final topology \( \tau \), especially \( \delta_{DEZ} \leq_c \rho \). But \( \delta_{DEZ} \) is not admissible since \( \rho \leq \delta_{DEZ} \). The names w.r.t. \( \delta_{DEZ} \) contain more finitely accessible information than those w.r.t. \( \rho \). A very undesirable consequence is that many trivial functions on \( IR \) are not even \((\delta_{DEZ}, \delta_{DEZ})\)-continuous.

Example: The function \( x \mapsto 3x \) is not \((\delta_{DEZ}, \delta_{DEZ})\)-continuous.
Consider the sequences \((p_n)\) and \((q_n)\) on \(\mathbb{N}\) defined by
\[p_n = (003\ldots 30\ldots), \quad q_n = (003\ldots 340\ldots)\] (each with \(n\)-times "3").
Suppose \(3 \cdot \delta_{DEZ}(r) = \delta_{DEZ}(r) (r \in \text{dom}(\delta_{DEZ}))\) for some \(r; \mathbb{N} \rightarrow \mathbb{N}\). Then \(\lim p_n = \lim q_n = (0033\ldots)\), but \(\lim r p(n) \neq \lim r q(n)\), hence \(r\) is not continuous. Therefore the decimal representation is not very useful for analysis, again for topological reasons.

Finally we consider the representations by characteristic functions of left cuts (or right cuts) which is also often considered. Let \(\nu\) be a total bijective numbering of some dense \((w.r.t. \tau_{\mathbb{R}})\) subset of \(\mathbb{R}\). Define \(\delta_{LV}\) by \(\text{dom}(\delta_{LV}) = \{p \in \mathbb{N}|(\exists x \in \mathbb{R})(\forall i) (\nu(i) = 0 \iff \nu(i) < x)\}\), \(\delta_{LV}(p) = \sup \nu p^{-1}(0)\) for \(p \in \text{dom}(\delta_{LV})\). Then \(\delta_{LV}\) is admissible and its final topology \(\tau_{LV}\) is generated by the basis \(\{(x,y)|x,y \in \mathbb{R}, x < y\}\). Especially, one easily shows \(\delta_{LV} \leq c_{\rho}\) and \(\tau_{LV} \leq \tau_{LV}\). Notice that the final topology \(\tau_{LV}\) depends on the dense subset \(S \subseteq \mathbb{R}\) which is numbered by \(\nu\).

Therefore no left cut representation \(\delta_{LV}\) of \(\mathbb{R}\) can be called "natural" for topological reasons. Right cut representations \(\delta_{RV}\) are defined accordingly. It should be noted that the representations defined by continued fractions is the infimum of \(\delta_{LV}\) and \(\delta_{RV}\) if \(\nu\) is some standard numbering of \(\mathbb{Q}\).

Among all the representations of \(\mathbb{R}\) we have considered, \(\rho\) is the only one which has the appropriate topological properties for the study of analysis. The special definition of \(\rho\) guarantees some positive computability properties which become important in complexity theory (Ko [10], Kreitz and Weihrauch [11]).
5. Compactness on $\mathbb{R}$

As an application of the theory of representations to constructive analysis we consider compactness on $\mathbb{R}$. We show that (at least) two different reasonable kinds of compactness can be formulated in our theory. Let $I$, defined by $I_{<j,k>}:=(v_0(j)-2^{-k};v_0(j)+2^{-k})$, be a standard numbering of a basis of $\tau_\mathbb{R}$. A standard representation $\omega$ of $\tau_\mathbb{R}$ is defined by $\omega(p):=\bigcup\{I_k|k \in \mathbb{M}_p\}$. The corresponding representation $\alpha$ of the closed subsets is $\alpha(p):=\mathbb{R}\setminus \omega(p)$. Another way specifying a closed set $A$ is to list all the open sets $I_k$ such that $A \cap I_k \neq \emptyset$. Let $\text{dom}(\alpha_c):=\{p|(\exists A, \text{closed}) \mathbb{M}_p = \{k|A \cap I_k \neq \emptyset\}\}$

$\alpha_c(p):=\bigcap_{k \geq 0} \{I_{<j,k>|<j,k> \in \mathbb{M}_p}\}$. The representations $\alpha$ and $\alpha_c$ are incomparable w.r.t. topological reducibility because the finitely accessible informations are too different. Both representations are admissible w.r.t. their final topologies. Let $\alpha_1$ be the standard infimum of $\alpha$ and $\alpha_c$. This representation corresponds to the concept of locatedness in constructive analysis (Bishop [2]). For a closed set $A,A=\{x|d(x,A)=0\}$, therefore the closed sets can be characterized by distance functions which are $(\rho,\rho)$-continuous. The $\check{\omega}$-names of the corresponding operators $F$ yield a representation $\alpha'$ of the nonempty closed subsets of $\mathbb{R}$ which directly expresses locatedness. The representation $\alpha_1$ (restricted to the nonempty sets) is equivalent to $\alpha'$. A subset of $\mathbb{R}$ is compact, iff it is closed and bounded. Therefore, the restriction $\check{\alpha}(\alpha_c,\alpha_1)$ of $\alpha(\alpha_c,\alpha_1)$ to the bounded subsets yields a representation of the compact subsets $K(\mathbb{R})$. Unfortunately sup:

$K(\mathbb{R}) \rightarrow \mathbb{R}$ is not $\check{\alpha}(\rho)-((\check{\alpha}_c,\rho), (\alpha_1,\rho))$ continuous,

that is, a $\rho$-name for sup (S) cannot be continuously obtained from an $\check{\alpha}$-name of S (etc.). However the following holds:

Theorem:

There are computable functions $\Sigma, \Gamma, \Delta: \mathbb{F} \rightarrow \mathbb{F}$ such that:
\[ \rho < \Sigma(p) = \sup \alpha_c(p) \text{ if } \alpha_c(p) \text{ is bounded,} \]
\[ \rho > \Gamma \langle i, p \rangle = \sup \alpha(p) \text{ if } \alpha(p) \in [-i; i], \]
\[ \rho \Delta \langle i, p \rangle = \sup \alpha_i(p) \text{ if } \alpha_i(p) \in [-i; i]. \]

**Proof:**

Since \( \mathcal{M}_p = \{ j \mid I_j \cap \alpha_c(p) \neq \emptyset \} \) for every \( p \in \text{dom } \alpha_c \), there is some computable \( \Sigma \in [\mathcal{F} \to \mathcal{F}] \) such that \( \mathcal{M}_{\Sigma(p)} = \{ i \mid v_D(i) < \sup \alpha_c(p) \} \) and therefore \( \rho < \Sigma(p) = \sup \alpha_c(p) \) whenever \( \alpha_c \) is bounded.

Now let \( p \in \text{dom } \alpha \), \( i \in \mathbb{N} \) such that \( \alpha(p) \in [-i; i] \). Then

\[ v_D(j) > \sup \alpha(p) \iff (v_D(j) > i \text{ or } [v_D(j); i] \in \mathbb{R} \setminus \alpha(p)) \]

Since \( \mathbb{R} \setminus \alpha(p) = \cup \{ I_k \mid k \in \mathcal{M}_p \} \) and \( [v_D(j); i] \) is compact in \( \mathbb{R} \), there is some computable \( \Gamma \in [\mathcal{F} \to \mathcal{F}] \) such that

\[ \mathcal{M}_\Gamma(p) = \{ j \mid v_D(j) > \sup \alpha(p) \} \text{ - i.e. } \rho > \Gamma(p) = \sup \alpha(p) - \]

whenever \( \alpha(p) \in [-i; i] \).

The last claim follows immediately from \( \alpha_c \in \text{Inf}_c \{ \alpha_c, \alpha \} \).

\[ \square \]

Therefore, for determining the supremum (continuously) it is not sufficient to know that \( \alpha_1(p) \) is bounded but a bound must be known (c.f. Bishop's concept of constructive compactness). The information of a bound can be inserted into the name. Define representations \( \alpha^b \) and \( \alpha^b_1 \) of the compact subsets of \( \mathbb{R} \) by \( \text{com}(\alpha^b) = \{ \langle i, p \rangle \mid \alpha(p) \in [-i; i] \} \), \( \alpha^b < i, p > := \alpha(p) \), and \( \alpha^b_1 \) correspondingly with \( \alpha_1 \). Then sup is \( (\alpha^b, \rho_>) \)-computable and \( (\alpha^b_1, \rho) \)-computable (but not \( (\alpha^b, \rho) \)-continuous). The compact
subsets of \( \mathbb{R} \) can also be characterized by their Heine-Borel property which leads to (at least) two different characterizations. Let \( C_p := \{ I_j | j \in \mathbb{M}_p \} \), \( C_p, n := \{ I_j | (3i \in \mathbb{D}_n) j+1 = p(i) \} \). Let us say "\( \mathbb{R} \) IF \( \rightarrow \) \( \mathbb{N} \) proves compactness of \( S \leq \mathbb{R} \)" iff

\[
(\forall p) [S \leq \bigcup C_p \iff p \in \text{dom}(\Omega) \text{ and } S \subseteq \bigcup C_p \implies S \subseteq \bigcup C_p, \Omega(p)].
\]

This induces a representation \( \kappa_w \) of the compact sets by \( \text{dom}(\kappa_w) = \{ p | \chi_p \text{ proves compactness of some } S \leq \mathbb{R} \} \), \( \kappa_w(p) = \) the set \( S \leq \mathbb{R} \) for which \( \chi_p \) proves compactness. Our first constructive version of the Heine-Borel theorem for \( \mathbb{R} \) is as follows:

**Theorem (Heine-Borel, weak)**

\[
\alpha^b = \omega \kappa_w
\]

**Proof:**

Let \( <i,p> \in \text{dom} \alpha^b \). Then \( \alpha^b<i,p> = [-i;i] \setminus \{ I_j | j \in \mathbb{M}_p \} \).

By compactness of \( [-i;i] \) we have

\[
\alpha^b<i,p> \subseteq \bigcup C_p \iff (\exists n) [-i;i] \leq \bigcup (C_{q,n} \cup C_{\mathbb{P},n})
\]

\[
= (\exists n) \alpha^b<i,p> \subseteq \bigcup C_{q,n}.
\]

Since \( <i;i> \leq \bigcup (C_{q,n} \cup C_{\mathbb{P},n}) \) is decidable there is some computable \( \Sigma \in [\mathbb{IF} \rightarrow \mathbb{IF}] \) with \( \alpha^b<i,p> = \kappa_w \Sigma<i,p> \) vor every \( <i,p> \in \text{dom} \alpha^b \).

Conversely \( \kappa_w(p) \in C_{\mathbb{M}} \) holds for every \( p \in \text{dom} \kappa_w \) (where \( \text{id}(n) = n \)) and since \( \bigcup C_{\mathbb{M},n} \subseteq [-i;i] \) is decidable, there is some computable \( \Delta \in [\mathbb{IF} \rightarrow \mathbb{IN}] \) with \( \kappa_w(p) \subseteq [-\Delta(p), \Delta(p)] \) if \( p \in \text{dom} \kappa_w \).

Furthermore \( \kappa_w(p) = \mathbb{R} \setminus \{ I_j \mid \overline{I_j} \cap \kappa_w(p) = \emptyset \} \) and

\[
\overline{I_j} \cap \kappa_w(p) = \emptyset \iff (\exists p \in \text{dom} \chi_p) \overline{I_j} \cap \bigcup C_{q}, \chi_p(q) = \emptyset
\]

(where \( \overline{I_j} \) denotes the closure of the interval \( I_j \)). Using the projection theorem (see [23]) one can construct some
computable $\Gamma \in [IF \to IF]$ with $M_{\Gamma}(p) = \{ j \mid \overline{\Gamma}_j \subseteq IR \setminus \kappa_w(p) \}$ especially $\omega \Gamma(p) = IR \setminus \omega \Gamma(p) = \kappa_w(p)$ - whenever $p \in \text{dom} \, \kappa_w$.

\[ \square \]

Since sup, therefore, is not $(\kappa_w, \rho)$-continuous, $\kappa_w$ is called the weak Heine-Borel representation. As we already know, the names $p$ w.r.t. $a_1^b$ contain more finitely accessible information. It can be shown that from such a name $p$ for any covering $C_p$ a minimal covering can be obtained. Let $\kappa$ be the restriction of $\kappa_w$ to those $p \in IF$ such that $C_q, \chi_p(q)$ is a minimal covering of $\kappa_w(p)$, i.e. $\kappa_w(p) \notin \cup E$ for any proper subset $E$ of $C_q, \chi_p(p)$. Then the (strong) Heine-Borel theorem can be formulated as follows:

**Theorem (Heine-Borel, strong)**

\[ a_1^b \leq_c \kappa \]

The proof is similar to that of the weak Heine-Borel theorem.

More details will be presented in a forthcoming paper [13].

6. Conclusion

In this contribution we have presented an approach to constructive analysis which is based on Type 1 and Type 2 recursion theory and on the theory of numberings and of representations. For the Type 2 theory, topology plays an important role. The approach extends the approach of the Polish school. Since for any representation $\delta$ there is a canonical numbering $\nu_\delta := \delta \Phi$ of the computable elements it includes the approach of the Russian school, and also essential ideas of the approaches like Bishop's can be expressed and studied without change of logic. We claim that this is the adequate way for investigating constructivity, computability and computational complexity in analysis (and other areas of mathematics).
The properties "continuous", "computable", "easily (e.g. polynomially) computable" yield a basic hierarchy of constructivity. It has turned out that in most cases a property which is not "effective" is not even continuous and a property which is "effective" is easily computable. This means that most of the negative results have topological reasons and are independent of Church's thesis. The fundamental role of topology in recursive analysis has already been pointed out by Nerode [18] and others.

The approach presented here does not depend on specific representations but different representations may be chosen from case to case. First of all a representation has to be topologically sound depending on the intended application. Then computability and computational complexity have to be considered. For any representation $\delta$ the corresponding numbering $\nu_\delta$ of the $\delta$-computable elements brings up the question how $\nu_\delta$-computability and $\delta$-computability are related. Complete or partial answers are known only for very few cases (Myhill and Shepherdson [17], Ceitin [5], Spreen [22]). Seemingly the clear topological properties of $\delta$ are concealed by the composition with the numbering $\varphi$ the topological properties of which are not easily to understand.

In this contribution we only gave some examples of application. It should be mentioned that the kind of computability introduced by Pour-El and Richards [19] on the $L^p$-spaces can very naturally be defined in our framework by use of representations. A general theory of constructive normed spaces can be developed. Even measure theory can easily be approached. By $d(X,Y) := \mu(X \Delta Y)$ a metric on the open sets $J_k := \bigcup \{ I_n \mid n \in D_k \}$ can be defined. The completion of this space is (essentially) the set of Borel sets factorized by the null sets.
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