A Matrix Characterization For Multiplicative Exponential Linear Logic [*]

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Abstract. We develop a matrix characterization of logical validity in $\mathcal{MELL}$, the multiplicative fragment of propositional linear logic with exponentials and constants. To prove the correctness and completeness of our characterization, we use a purely proof theoretical justification rather than semantical arguments.

Our characterization is based on concepts similar to matrix characterizations proposed by Wallen for other non-classical logics. It provides a foundation for developing proof search procedures for $\mathcal{MELL}$ by adopting techniques that are based on these concepts and also makes it possible to adopt algorithms that translate the machine-found proofs back into the usual sequent calculus for $\mathcal{MELL}$.

Keywords: Linear Logic, Automated Deduction, Connection Method

1. Introduction

Linear logic (Girard, 1987) is an expressive formalism for reasoning about action and change. Formulas are considered resources that disappear after their use unless explicitly marked as reusable. *Frame axioms* about the environment (McCarthy and Hayes, 1969) do not have to be stated. One only has to deal with axioms about objects involved in the action. Linear logic has found many useful applications such as logic programming (Hodas and Miller, 1994; Miller, 1996), modeling concurrent computation (Gehlot and Gunter, 1991), planning (Masseron et al., 1991), and other areas.

The expressiveness of linear logic, however, results in a high complexity. Propositional linear logic is undecidable. The multiplicative fragment ($\mathcal{MLL}$) is already $\mathcal{NP}$-complete (Lincoln and Winkler, 1994). Whether the multiplicative exponential fragment ($\mathcal{MELL}$) is decidable or not is still unknown. Consequently, proof search in linear logic is difficult to automate. Girard's original sequent calculus (Girard, 1987) contains too many redundancies to be useful for efficient proof search. Attempts to remove permutabilities from sequent proofs (Andreoli,

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1993; Galmiche and Perrier, 1994) and to add proof strategies (Tammet, 1994) have provided significant improvements, but some redundancies remain. Proof nets (Danos and Regnier, 1989), on the other hand, can handle only a fragment of the logic. To handle the remaining parts one has to introduce the concept of boxes (Girard, 1987), which again cause major problems for automated proof search. Although there has been progress in removing some boxes (Girard, 1996), efficient proof search for full linear logic appears to be beyond the scope of proof nets.

Matrix characterizations of logical validity, originally developed as foundation of the *connection method* for classical logic (Andrews, 1981; Bibel, 1981; Bibel, 1987), can be understood as compact representations of the search space for tableaux, natural deduction, or sequent calculi. They avoid the redundancies contained in these calculi and are driven by *complementary connections* between atomic formulae of different *polarity*, i.e. possible leaves in a sequent proof, instead of the logical connectives of a proof goal.

Wallen has extended the classical matrix characterization to intuitionistic and modal logics (Wallen, 1990; Waaler, 2001). While in classical logic two connected atomic formulae are complementary if their subterms can be unified, in non-classical logics also the *prefixes* of the two atoms, i.e. strings describing their position in the formula tree, must be unifiable. Differences between logics are encoded by different rules for constructing these prefixes from *special positions* that are inserted into the formula tree and different requirements on the unifiers. As there are efficient algorithms for prefix unification (Otten and Kreitz, 1996) these characterizations serve as a basis for a uniform proof search method (Kreitz and Otten, 1999) and a method for translating matrix proofs into sequent proofs (Kreitz and Schmitt, 2000).

Resource management similar to $\mathcal{MLL}$ is addressed by the *linear connection method* (Bibel, 1986). In (Kreitz et al., 1997) we have developed a matrix characterization for $\mathcal{MLL}$ that extends Wallen's approach and extended the proof search and translation procedures from (Kreitz and Otten, 1999; Kreitz and Schmitt, 2000) accordingly. Fronhöfer (Fronhöfer, 1996) gives a matrix characterization of $\mathcal{MLL}$ that captures some aspects of weakening and contraction. Galmiche's matrix characterizations for fragments of linear logic (Galmiche, 2000) are based on acyclic connection graphs. Both approaches are closely related to proof nets and do not appear to generalize any further. In fact, (Fronhöfer, 1996, page 255) states that it may not be possible to develop matrix characterizations for larger fragments of linear logic.

> *Since no good characterization of the resulting fragment [adding multiplicative constants to* $\mathcal{MLL}$*] of linear logic by means of proof nets is known, we are sceptical that a nice matrix theory exists.*

In this article we will show that it *is* possible to develop a matrix characterization for $\mathcal{MELL}$, the multiplicative fragment of linear logic together with exponentials and the multiplicative constants **1** and $\perp$.

Rather than following Wallen's *semantical* approach to developing a matrix characterization of logical validity, we will use a purely *proof-theoretical* one. Starting with a calculus Σ_1 that adopts Smullyan's uniform tableaux notation to Girard's sequent calculus for $\mathcal{MELL}$ we apply a series of syntactical transformations that make the search space of a proof calculus more compact. In each step, the resulting proof calculus is proven correct and complete with respect to the previous one. By using this methodology the development of the technically quite demanding matrix characterization for $\mathcal{MELL}$ becomes manageable.

Our approach uses Andreoli's focusing principle (Andreoli, 1993; Andreoli, 2001) in one of its first design steps. While this is not necessary for developing the matrix characterization, it is a valuable optimization, leading to a more compact matrix representation of logical formulas.

This article is organized as follows. In Section 2 we give a brief introduction to $\mathcal{MELL}$ and the calculus Σ_1, introduce the concept of multiplicities, and define a *triadic calculus* Σ_3' that is closely related to Andreoli's calculus Σ_3 (Andreoli, 1993) and serves as a first step in the development of our matrix characterization. A reader familiar with linear logic and Andreoli's work may proceed directly to Section 3, where we introduce the concept of matrices for $\mathcal{MELL}$ formulas. Section 4 defines a sequent calculus Σ_{pos} for matrices as second step in our derivation. In Section 5 we finally arrive at the matrix characterization for $\mathcal{MELL}$, which we prove sound and complete in Section 6. We conclude in Section 7 with a discussion of related work and possible future extensions and applications of our results. To keep this article self-contained, we present a complete derivation of Σ_3' in Appendix A.

2. Propositional Linear Logic

Linear Logic (Girard, 1987) is a resource sensitive logic. Proof theoretically it can be seen as the outcome of removing the rules for contraction and weakening from classical sequent calculus and re-introducing them in a controlled manner. Linear negation $^\perp$ is involutive like classical negation. Two different traditions for writing the sequent rule for classical conjunction result in two different conjunctions $\otimes$ and $\&$ and in two different disjunctions $⅋$ and $\oplus$. For the same reason the constant `true` splits up into **1** and $\top$ and `false` splits up into $\perp$ and **0**. The unary connectives ? and ! allow a controlled application of weakening and contraction. Quantifiers $\forall$ and $\exists$ can be added as in classical logic.

2.1. Syntax

As usual, the syntax of linear logic is determined by the concepts of formulas, sub-formulas and formula trees. The *formulas* of propositional linear logic are defined recursively.

- Each $A \in \mathcal{P}$ is a formula, where $\mathcal{P}$ is a set of basic propositions.
- $\mathbf{1}$, $\bot$, $\top$, and $\mathbf{0}$ are formulas.
- ${F_1}^{\bot}$, $F_1 \otimes F_2$, $F_1 \parr F_2$, $F_1 \multimap F_2$, $F_1 \& F_2$, $F_1 \oplus F_2$, $!F_1$, and $?F_1$ are formulas if F_1 and F_2 are formulas.

Intuitively, linear logic formulas can be understood as resources that are consumed during the production of other resources. Basic propositions in $\mathcal{P}$ correspond to primitive resources. A primitive resource $\mathbf{1}$ can be consumed without producing any other resource and be produced if no other resource is present. When $\top$ is produced arbitrary resources can be consumed or produced, while the consumption of $\top$ is impossible.

The connectives of linear logic construct complex resources. To produce $F_1 \otimes F_2$, F_1 and F_2 must be produced under *sharing* of resources: a resource consumed in the construction of F_1 cannot be consumed in the construction of F_2 and vice versa. To consume $F_1 \otimes F_2$, both F_1 and F_2 must be consumed. To produce $F_1 \& F_2$, F_1 and F_2 must be produced under *duplication* of resources: duplicates of all resources consumed in the construction of F_1 must be consumed in the construction of F_2 and vice versa. To consume $F_1 \& F_2$ one may choose which of the two shall be consumed. $!F_1$ acts like a machine that produces an arbitrary number of resources when it is used up. To construct $!F_1$, one must be able to construct an arbitrary number of duplicates of F_1.

Linear implication $F_1 \multimap F_2$ specifies a state transition during which F_1 is consumed and F_2 is produced. It is equivalent to $(F_1 \otimes {F_2}^{\bot})^{\bot}$. Linear negation $^{\bot}$ expresses the differences between the consumption and the production of resources: F_1 must be consumed to produce ${F_1}^{\bot}$ and produced to consume ${F_1}^{\bot}$. The meaning of the constants $\bot$ and $\mathbf{0}$, the disjunctions $\parr$ and $\oplus$, and the exponential $?$ is explained by the following equivalences:

$$\begin{array}{lcl@{\qquad\qquad}lclcl}
F & \equiv & F^{\bot\bot} & F_1 \parr F_2 & \equiv & ({F_1}^{\bot} \otimes {F_2}^{\bot})^{\bot} & \equiv & {F_1}^{\bot} \multimap F_2 \\
\bot & \equiv & \mathbf{1}^{\bot} & F_1 \oplus F_2 & \equiv & ({F_1}^{\bot} \& {F_2}^{\bot})^{\bot} & & \\
\mathbf{0} & \equiv & \top^{\bot} & ?F_1 & \equiv & (!({F_1}^{\bot}))^{\bot} & &
\end{array}$$

Linear logic can be divided into the multiplicative, additive, and exponential fragment. $\mathbf{1}$, $\bot$, $\otimes$, $\parr$, and $\multimap$ are the connectives of the *multiplicative fragment*, in which resources are used exactly once. Resource sharing is enforced in the *additive fragment*, whose connectives are $\top$, $\mathbf{0}$, $\&$, and $\oplus$. By means of the *exponentials* $!$ and $?$ formulas are

Table I. Uniform notation for signed $\mathcal{MELL}$ formulas

lit	$\langle A, -\rangle$	$\langle A, +\rangle$	
τ	$\langle \bot, -\rangle$	$\langle \mathbf{1}, +\rangle$	
ω	$\langle \mathbf{1}, -\rangle$	$\langle \bot, +\rangle$	
ν	$\langle !F, -\rangle$	$\langle ?F, +\rangle$	
$succ_1(\nu)$	$\langle F, -\rangle$	$\langle F, +\rangle$	
π	$\langle ?F, -\rangle$	$\langle !F, +\rangle$	
$succ_1(\pi)$	$\langle F, -\rangle$	$\langle F, +\rangle$	
o	$\langle F^{\bot}, -\rangle$	$\langle F^{\bot}, +\rangle$	
$succ_1(o)$	$\langle F, +\rangle$	$\langle F, -\rangle$	
α	$\langle F_1 \otimes F_2, -\rangle$	$\langle F_1 \parr F_2, +\rangle$	$\langle F_1 \multimap F_2, +\rangle$
$succ_1(\alpha)$	$\langle F_1, -\rangle$	$\langle F_1, +\rangle$	$\langle F_1, -\rangle$
$succ_2(\alpha)$	$\langle F_2, -\rangle$	$\langle F_2, +\rangle$	$\langle F_2, +\rangle$
β	$\langle F_1 \otimes F_2, +\rangle$	$\langle F_1 \parr F_2, -\rangle$	$\langle F_1 \multimap F_2, -\rangle$
$succ_1(\beta)$	$\langle F_1, +\rangle$	$\langle F_1, -\rangle$	$\langle F_1, +\rangle$
$succ_2(\beta)$	$\langle F_2, +\rangle$	$\langle F_2, -\rangle$	$\langle F_2, -\rangle$

marked as being reusable. All fragments can be combined freely and exist on their own right. From now on we will focus on $\mathcal{MELL}$, the combination of the multiplicative fragment with the exponentials.

As usual, the *sub-formulas* of a formula F can be determined recursively. The *major sub-formula* of a formula $F^{\bot}$, $?F$, or $!F$ is F. The major sub-formulas of $F_1 \otimes F_2$, $F_1 \parr F_2$, or $F_1 \multimap F_2$ are F_1 and F_2. We define $succ_1$ and $succ_2$ as functions that return the major sub-formulas of a given formula, e.g., $succ_1(F_1 \otimes F_2) = F_1$ and $succ_2(F_1 \otimes F_2) = F_2$. $succ_2$ is undefined for formulas $F^{\bot}$, $?F$, or $!F$. Both functions are undefined for atomic formulas and for constants. $succ_1$ and $succ_2$ induce a relation $\prec$ on the sub-formulas of a formula: $F \prec G$ holds if G is a major sub-formula of F, i.e. if $G = succ_1(F)$ or if $G = succ_2(F)$. The transitive closure of $\prec$ is an ordering that we denote by $\ll$.

The *formula tree* of a formula F is the representation of $\prec$ as a graph. It has a node for each occurrence of a sub-formula of F while edges connect sub-formulas of F with their major sub-formulas.

To achieve a compact representation we adopt Smullyan's concept of *signed formulas* for $\mathcal{MELL}$. A *signed formula* $\langle F, k\rangle$ relates a formula F, its *label*, to a *polarity* $k \in \{+, -\}$. Depending on the label and on the polarity, a signed formula is associated with a *type* from lit, o, τ, ω, α, β, ν, and π according to Table I. This table also defines the extension of the functions $succ_1$ and $succ_2$ to signed formulas. Note that the polarity switches only for sub-formulas of $^{\bot}$ and $\multimap$. Often, we will use type symbols as meta-variables for signed formulas of the respective type,

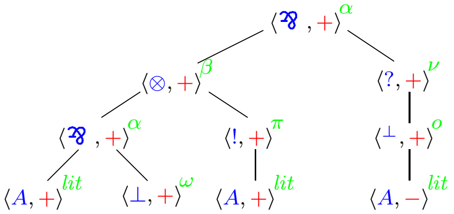

Figure 1. Formula tree for $F_1 = ((A \parr \perp) \otimes !A) \parr ?(A^{\perp})$

e.g., α stands for a signed formula of type α. Similarly, type symbols in boldface represent multi-sets of signed formulas of the respective type. As an example, the formula tree for $F_1 = ((A \parr \perp) \otimes !A) \parr ?(A^{\perp})$, together with types and polarities is presented in Figure 1.

2.2. A Sequent Calculus for $\mathcal{MELL}$

IDENTITY

$$\frac{}{\langle A, + \rangle, \langle A, - \rangle}\ axiom$$

TRANSITIVITY

$$\frac{\Upsilon_1, \langle F, - \rangle \quad \Upsilon_2, \langle F, + \rangle}{\Upsilon_1, \Upsilon_2}\ cut$$

NEGATION

$$\frac{\Upsilon, succ_1(o)}{\Upsilon, o}\ o$$

MULTIPLICATIVE FRAGMENT

$$\frac{}{\tau}\ \tau \qquad \frac{\Upsilon}{\Upsilon, \omega}\ \omega \qquad \frac{\Upsilon, succ_1(\alpha), succ_2(\alpha)}{\Upsilon, \alpha}\ \alpha \qquad \frac{\Upsilon_1, succ_1(\beta) \quad \Upsilon_2, succ_2(\beta)}{\Upsilon_1, \Upsilon_2, \beta}\ \beta$$

EXPONENTIAL FRAGMENT

$$\frac{\Upsilon}{\Upsilon, \nu}\ w \qquad \frac{\Upsilon, \nu, \nu}{\Upsilon, \nu}\ c \qquad \frac{\Upsilon, succ_1(\nu)}{\Upsilon, \nu}\ \nu \qquad \frac{\boldsymbol{\nu}, succ_1(\pi)}{\boldsymbol{\nu}, \pi}\ \pi$$

Figure 2. The unary sequent calculus Σ_1 for $\mathcal{MELL}$

Sequent calculi are usually described by rules that operate on sets of formulas. However, for a resource-sensitive logic it is more natural to consider *multi-sets*, because they allow a distinction between multiple occurrences of the same type of resource. The sequenct calculus Σ_1, depicted in Figure 2, operates on *unary sequents*, i.e. sequents that consist of a single multi-set Υ of signed formulas. Apart from the use of types instead of logical connectives, Σ_1 is equivalent to Girard's original calculus (Girard, 1987). In particular, the soundness, completeness, and cut-elimination results from (Girard, 1987) can be directly transfered to Σ_1: Σ_1 is sound, complete, and enjoys the cut-elimination property.

EXAMPLE 1. A Σ_1-proof of formula F_1 from Figure 1 is given below.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\langle A,+\rangle,\langle A,-\rangle}\,\mathit{axiom}
}{\langle A,+\rangle,\langle A^{\perp},+\rangle}\,o
}{\langle A,+\rangle,\langle ?(A^{\perp}),+\rangle}\,\nu
}{\langle A,+\rangle,\langle \perp,+\rangle,\langle ?(A^{\perp}),+\rangle}\,\omega
}{\langle A\parr\perp,+\rangle,\langle ?(A^{\perp}),+\rangle}\,\alpha
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\langle A,+\rangle,\langle A,-\rangle}\,\mathit{axiom}
}{\langle A,+\rangle,\langle A^{\perp},+\rangle}\,o
}{\langle A,+\rangle,\langle ?(A^{\perp}),+\rangle}\,\nu
}{\langle !A,+\rangle,\langle ?(A^{\perp}),+\rangle}\,\pi
}{
\dfrac{
\dfrac{\langle (A\parr\perp)\otimes !A,+\rangle,\langle ?(A^{\perp}),+\rangle,\langle ?(A^{\perp}),+\rangle}{\langle (A\parr\perp)\otimes !A,+\rangle,\langle ?(A^{\perp}),+\rangle}\,c
}{\langle ((A\parr\perp)\otimes !A)\parr ?(A^{\perp}),+\rangle}\,\alpha
}\,\beta
\qquad \diamond
$$

As usual, we refer to the sequents above the line in a rule as the *premises* and to the sequent below as the *conclusion*. A *principal formula* is a formula that occurs in the conclusion but not in any premise. Formulas that occur in a premise but not in the conclusion are called *active*. All other formulas compose the *context*. For instance, the calculus rule π demands that all context formulas have type ν. Girard's cut-elimination result allows us to restrict our considerations to proofs in which the cut rule is not applied. Such proofs have the *sub-formula property*, i.e. all formulas that occur in a premise of a rule application also occur in the conclusion. This property is essential for matrix characterizations. We further restrict our considerations to proofs that are *normal* in the sense that the contraction rule is applied only in certain places.

DEFINITION 2. A signed formula ν is *contracted* in a proof $\mathcal{P}$ if the contraction rule c is applied on ν in $\mathcal{P}$. A proof $\mathcal{P}$ for a unary sequent Υ is in *contraction normal form* (briefly *contraction normal*) if one of the following conditions holds.

1. $\mathcal{P}$ consists only of an application of *axiom* or τ.
2. $\mathcal{P}$ results from a contraction normal proof $\mathcal{P}'$ for a sequent Υ' by applying one of the rules o, ω, α, ν, or w with Υ' as premise.
3. $\mathcal{P}$ results from contraction normal proofs $\mathcal{P}'$ and $\mathcal{P}''$ for sequents Υ' and Υ'' by applying the rule β with Υ' and Υ'' as premises.
4. $\mathcal{P}$ results from a contraction normal proof $\mathcal{P}'$ for a sequent Υ' by subsequent applications of c to the same occurrence of a signed formula ν in Υ' and neither ν nor any copy of ν is contracted in $\mathcal{P}'$.
5. $\Upsilon = \boldsymbol{\nu}, \pi$ and $\mathcal{P}$ results from a contraction normal proof $\mathcal{P}'$ for a sequent Υ' by applying the rule π with premise Υ' and no formula $\nu \in \boldsymbol{\nu}$ is contracted in $\mathcal{P}'$.

A restriction to proofs in contraction normal form does not affect completeness. If a unary sequent Υ is provable in Σ_1 then there also is a contraction normal proof $\mathcal{P}'$ for Υ. This property follows from results in (Galmiche and Perrier, 1994), which state that applications of the contraction rule can be moved towards the root of a proof.

Proof search in sequent calculi is often performed in an analytic fashion, i.e. one starts with the sequent to be proven and reduces it by a reverse application of rules until all branches of the proof tree have been closed by rules that have no premises. In Σ_1, the rules *axiom* or τ can be used to close branches of a proof tree. There are several choice points during *analytic proof search* in Σ_1. Firstly, a principal formula must be chosen. This choice already determines which rule must be applied unless the principal formula has type ν. For formulas of type ν, one has to choose between the rules w, c, and ν. Secondly, if a β-rule is applied, the context of the sequent must be partitioned onto the premises (*context splitting*).[1] Additional difficulties arise from the rules *axiom*, τ, and π. *axiom* and τ require an empty context, which expresses that all formulas must be used up in a proof. The π rule requires that all formulas in the context are of type ν. Finally, the rules β and π give rise to *non-permutabilities*. While subsequent applications of many rules can be permuted freely within a proof, the application of β and π *affects the context* of a formula and cannot be permuted with other rule applications.[2] Though the connectives of linear logic make proof search more difficult they also give rise to new possibilities. Some applications for linear logic programming are illustrated in (Miller, 1996).

2.3. EXPLICATING MULTIPLICITIES

In sequent proofs there are two possible notions of *occurrence* of a formula φ: an occurrence of φ as sub-formula in some formula tree or its occurrences within a derivation. The difference between these two becomes only apparent when contraction is applied in order to increase the number of occurrences of φ in the sequent during analytical proof search. In $\mathcal{MELL}$, contraction may only be applied to formulas of type ν, i.e. only ν-formulas are *generic*. Usually, it is rather difficult to decide when the contraction rule shall be applied. A proof attempt may fail if there are not enough copies of a formula. But in general it is impossible to calculate an upper bound for the number of copies required, which means that proof search methods have to increase this number dynamically. A compact representation of proof search

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[1] These choice points are not specific to Σ_1. Similar choices have to be made during proof search with Girard's original sequent calculus and several solutions have been proposed in order to optimize them (Andreoli, 1993; Andreoli, 2001; Galmiche and Perrier, 1994; Tammet, 1994; Cervesato et al., 2000; Harland and Pym, 1997).

[2] For a more detailed discussion of the permutability of rules for linear logic, we refer to (Galmiche and Perrier, 1994).

– like the matrix characterization we are aiming for – must therefore somehow incorporate the number of copies of generic formulas. Usually, multiplicity functions are employed for this purpose.

A *multiplicity function* μ for a signed formula φ returns for each sub-formula ν of φ a natural number n. n is called the *multiplicity* of ν. The multiplicity $\mu(\varphi)$ will be used to encode the number of copies of φ that are used in a sequent proof. The formula tree for F_1 in Figure 1, for instance, has one ν-node, corresponding to the sub-formula $?(A^\perp)$. In the Σ_1-proof for F_1 in Example 1, this formula is needed and the contraction rule had to be applied to it once. Thus, $\mu(?(A^\perp))$=2.

The use of multiplicities allows us to define the following calculus rule that reduces a formula ν to $\mu(\nu)$ copies of ν for a given function μ.

$$\frac{\Upsilon, \nu^{\mu(\nu)}}{\Upsilon, \nu}\ c_\mu$$

For applications of c_μ, the following *side condition* must hold: in a given proof, c_μ may not be applied to reduce a formula of type ν that is already a principal or active formula of another application of c_μ. I.e. c_μ may be applied at most once on a given formula, including copies thereof. This constraint can be easily implemented by introducing tags on formulas where initially all formulas are untagged, principal and active formulas in each application of c_μ are tagged, and an application of c_μ is only allowed if the principal formula is *not* tagged.

Note that an application of c_μ has the same effect as an application of the rule w if $\mu(\nu) = 0$, as no rule application if $\mu(\nu) = 1$, and as $\mu(\nu) - 1$ applications of rule c if $\mu(\nu) > 1$. Consequently, adding this rule to the calculus Σ_1 does not affect soundness. After adding c_μ to Σ_1, contraction normal proofs can be carried out without applying any of the rules c or w (by using c_μ and choosing μ appropriately). Since a restriction to proofs that are in contraction normal form does not affect completeness (cf. Section 2.2), we may safely remove these rules from the calculus. The calculus Σ_1' that results from Σ_1 by adding c_μ and by removing c and w is sound as well as complete.

2.4. A Variant of Andreoli's Focusing Principle

Although it is possible to derive a matrix characterization for $\mathcal{MELL}$ directly from Σ_1', we aim at a more compact matrix characterization. Therefore, we derive the characterization from a calculus that causes less redundancies during proof search. We introduce a variant of Andreoli's triadic sequent calculus Σ_3 (Andreoli, 1993), which compactifies the search space by introducing the *focusing principle* (Andreoli, 2001). Our calculus Σ_3' differs from Andreoli's calculus in the way structural

IDENTITY

$$\frac{}{\Theta : \langle A, +\rangle, \langle A, -\rangle \Uparrow \cdot}\ \mathit{axiom}$$

NEGATION

$$\frac{\Theta : \Upsilon \Downarrow \mathit{succ}_1(o)}{\Theta : \Upsilon \Downarrow o}\ o{\Downarrow} \qquad \frac{\Theta : \Upsilon \Uparrow \Xi, \mathit{succ}_1(o)}{\Theta : \Upsilon \Uparrow \Xi, o}\ o{\Uparrow}$$

MULTIPLICATIVE FRAGMENT

$$\frac{}{\Theta : \tau \Uparrow \cdot}\ \tau \qquad \frac{\Theta : \Upsilon \Uparrow \Xi}{\Theta : \Upsilon \Uparrow \Xi, \omega}\ \omega$$

$$\frac{\Theta : \Upsilon \Uparrow \Xi, \mathit{succ}_1(\alpha), \mathit{succ}_2(\alpha)}{\Theta : \Upsilon \Uparrow \Xi, \alpha}\ \alpha \qquad \frac{\Theta_1 : \Upsilon_1 \Downarrow \mathit{succ}_1(\beta) \quad \Theta_2 : \Upsilon_2 \Downarrow \mathit{succ}_2(\beta)}{\Theta_1, \Theta_2 : \Upsilon_1, \Upsilon_2 \Downarrow \beta}\ \beta$$

EXPONENTIAL FRAGMENT

$$\frac{\Theta, \mathit{succ}_1(\nu)^{\mu(\nu)} : \Upsilon \Uparrow \Xi}{\Theta : \Upsilon \Uparrow \Xi, \nu}\ \nu \qquad \frac{\Theta : \cdot \Uparrow \mathit{succ}_1(\pi)}{\Theta : \cdot \Downarrow \pi}\ \pi$$

FOCUSING

$$\frac{\Theta : \Upsilon \Downarrow \varphi}{\Theta, \varphi : \Upsilon \Uparrow \cdot}\ \mathit{focus}_1 \qquad \frac{\Theta : \Upsilon \Downarrow \varphi}{\Theta : \Upsilon, \varphi \Uparrow \cdot}\ \mathit{focus}_2{}^{*}$$

$$\frac{\Theta : \Upsilon, \varphi \Uparrow \Xi}{\Theta : \Upsilon \Uparrow \Xi, \varphi}\ \mathit{defocus}^{**} \qquad \frac{\Theta : \Upsilon \Uparrow \varphi}{\Theta : \Upsilon \Downarrow \varphi}\ \mathit{switch}^{***}$$

$*$ In *focus*$_2$ φ must not be of type *lit* or τ.
$**$ In *defocus* φ must be of type *lit*, τ, β, or π.
$***$ In *switch* φ must be of type *lit*, τ, ω, α, or ν.

Figure 3. The triadic sequent calculus Σ_3' for $\mathcal{MELL}$

rules are handled. While Andreoli uses a lazy strategy for contraction and weakening, our calculus is based on multiplicities, which results in an eager handling of contraction and a lazy handling of weakening.

As depicted in Figure 3, Σ_3' operates on *triadic sequents*, i.e. multi-sets of signed formulas where each formula is associated with one of three zones: an *unbounded zone*, a *bounded zone*, or a *focused zone*. Moreover, a triadic sequent is tagged with one of two modes: a *synchronous mode* or an *asynchronous mode*. Formally, a triadic sequent in synchronous mode has the form $\Theta : \Upsilon \Downarrow \varphi$ where Θ and Υ are multi-sets of signed formulas and φ is a single signed formula. A triadic sequent in asynchronous mode has the form $\Theta : \Upsilon \Uparrow \Xi$ where Θ and Υ are multi-sets of signed formulas and Ξ is a (possibly empty) sequence of signed formulas. In a triadic sequent, Θ constitutes the *unbounded zone*, Υ the *bounded zone*, and φ or Ξ the *focused zone*.

The fact that a given formula is in a particular zone encodes information about this formula. The rules of Σ_3' ensure that all formulas in the unbounded zone have originated from ν-formulas: they have been major sub-formulas of generic formulas in some sequent of the proof (hence the name *unbounded*). This requirement is not imposed on formulas in the bounded zone. To reduce a formula in the triadic calculus, it must be in the focused zone. This is the basis of the focusing principle, which allows one to fix the order of permutable rules without losing completeness. Focusing enforces a certain order of rule applications after a formula has been moved into the focus and, thereby, reduces non-determinism in proof-search.

There are some obvious differences between Σ_3' and Σ_1'. Σ_3' operates on triadic instead of unary sequents. Moreover, Σ_3' has four rules ($focus_1$, $focus_2$, *defocus*, and *switch*) that do not have a counterpart in Σ_1'. The purpose of these rules is to move formulas from one zone to another one or to change the mode of a triadic sequent.

Proof search in Σ_3' proceeds as follows: $focus_1$ or $focus_2$ is applied to move a formula from the unbounded or the bounded zone into the focus and to switch the mode to $\Downarrow$. These rules are only applicable if the focus is empty.[3] After one has focused a particular formula, this formula may be reduced by applications of the rules $o\Downarrow$, β, and π until none of these rules is applicable. Afterwards, the rule *switch* needs to be applied to switch the mode of the sequent from $\Downarrow$ to $\Uparrow$. Mode switching also occurs as side effect of rule π. In mode $\Uparrow$, the last formula in the focus is reduced by applications of $o\Uparrow$, ω, α, or ν until none of these rule is applicable. Afterwards, the last formula in the focus can be moved into the bounded zone by an application of *defocus*. Defocusing the last formula in the focus is also a side effect of rule ν. However, in this case the formula is moved into the unbounded zone. This process continues until no formulas are left in the focus. At this point the branch may be closed by one of the rules *axiom* or τ (if applicable) or, alternatively, another formula can be moved into the focus and so on. There are a few more technical differences between the two calculi:

- The rules *axiom* and τ of Σ_3' may be applied even if the context is not empty. However, only the unbounded zone may be nonempty.
- The ν-rule of Σ_3' does not correspond to a single rule in Σ_1' but rather to a combination of the rules c_μ and ν.
- While the π-rule in Σ_3' requires *all* context formulas to be in the unbounded zone, the π-rule in Σ_1' requires *all* context formulas to be of type ν.

Thus, the focusing principle determines a reduction ordering for *layers of formulas* rather than for individual formulas, which means

---

[3] We denote empty sequence of signed formulas by $\cdot$.

that less permutations of rule applications have to be considered during proof search. Nevertheless, our calculus Σ_3' is not intended for sequent calculus-based proof search but only as a step towards the matrix characterization, which leads to an even more compact search space. As in Σ_1', derivations are defined with respect to a fixed multiplicity μ. A signed formula φ is *derivable in* Σ_3' if $\cdot : \cdot \Uparrow \varphi$ is derivable for some μ.

EXAMPLE 3. We prove the signed formula $\langle F_1, +\rangle$ from Figure 1 in Σ_3'. Note that in the proof the multiplicity for the ν-rule is 2.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{axiom}
\UnaryInfC{$\cdot : \langle A,+\rangle, \langle A,-\rangle \Uparrow \cdot$}
\RightLabel{$\mathit{defocus}$}
\UnaryInfC{$\cdot : \langle A,+\rangle \Uparrow \langle A,-\rangle$}
\RightLabel{switch}
\UnaryInfC{$\cdot : \langle A,+\rangle \Downarrow \langle A,-\rangle$}
\RightLabel{$o \Downarrow$}
\UnaryInfC{$\cdot : \langle A,+\rangle \Downarrow \langle A^{\perp},+\rangle$}
\RightLabel{focus_1}
\UnaryInfC{$\langle A^{\perp},+\rangle : \langle A,+\rangle \Uparrow \cdot$}
\RightLabel{$\mathit{defocus}$}
\UnaryInfC{$\langle A^{\perp},+\rangle : \cdot \Uparrow \langle A,+\rangle$}
\RightLabel{ω}
\UnaryInfC{$\langle A^{\perp},+\rangle : \cdot \Uparrow \langle A,+\rangle, \langle \perp,+\rangle$}
\RightLabel{α}
\UnaryInfC{$\langle A^{\perp},+\rangle : \cdot \Uparrow \langle A \parr \perp,+\rangle$}
\RightLabel{switch}
\UnaryInfC{$\langle A^{\perp},+\rangle : \cdot \Downarrow \langle A \parr \perp,+\rangle$}
\AxiomC{}
\RightLabel{axiom}
\UnaryInfC{$\cdot : \langle A,+\rangle, \langle A,-\rangle \Uparrow \cdot$}
\RightLabel{$\mathit{defocus}$}
\UnaryInfC{$\cdot : \langle A,+\rangle \Uparrow \langle A,-\rangle$}
\RightLabel{switch}
\UnaryInfC{$\cdot : \langle A,+\rangle \Downarrow \langle A,-\rangle$}
\RightLabel{$o \Downarrow$}
\UnaryInfC{$\cdot : \langle A,+\rangle \Downarrow \langle A^{\perp},+\rangle$}
\RightLabel{focus_1}
\UnaryInfC{$\langle A^{\perp},+\rangle : \langle A,+\rangle \Uparrow \cdot$}
\RightLabel{$\mathit{defocus}$}
\UnaryInfC{$\langle A^{\perp},+\rangle : \cdot \Uparrow \langle A,+\rangle$}
\RightLabel{π}
\UnaryInfC{$\langle A^{\perp},+\rangle : \cdot \Downarrow \langle !A,+\rangle$}
\RightLabel{β}
\BinaryInfC{$\langle A^{\perp},+\rangle, \langle A^{\perp},+\rangle : \cdot \Downarrow \langle (A \parr \perp) \otimes !A,+\rangle$}
\RightLabel{focus_2}
\UnaryInfC{$\langle A^{\perp},+\rangle, \langle A^{\perp},+\rangle : \langle (A \parr \perp) \otimes !A,+\rangle \Uparrow \cdot$}
\RightLabel{$\mathit{defocus}$}
\UnaryInfC{$\langle A^{\perp},+\rangle, \langle A^{\perp},+\rangle : \cdot \Uparrow \langle (A \parr \perp) \otimes !A,+\rangle$}
\RightLabel{ν}
\UnaryInfC{$\cdot : \cdot \Uparrow \langle (A \parr \perp) \otimes !A,+\rangle, \langle ?(A^{\perp}),+\rangle$}
\RightLabel{α}
\UnaryInfC{$\cdot : \cdot \Uparrow \langle ((A \parr \perp) \otimes !A) \parr ?(A^{\perp}),+\rangle$}
\end{prooftree}
$$

◇

Σ_3' is sound and complete in the sense that a signed formula φ is provable in Σ_3' if and only if it is derivable in Σ_1'.

THEOREM 4 (Soundness and Completeness of $\mathbf{\Sigma_3'}$). *A signed formula* φ *is derivable in* Σ_3' *if and only if it is derivable in* Σ_1.

Theorem 4 can be proved along the same lines as Andreoli's soundness and completeness results for his calculus Σ_3. However, due to slight modifications of the calculus, there are a few technical differences in the proofs. A detailed proof of Theorem 4 can be found in Appendix A.

3. Matrices for $\mathcal{MELL}$

In this section we will introduce matrices as representations of formulas that will lead to very compact search spaces. The basic concepts of matrices are motivated informally in Section 3.1 before we define matrices for $\mathcal{MELL}$ formally in Section 3.2.

3.1. Towards Matrices for $\mathcal{MELL}$

Matrix-based proof search is based on path exploration. To prove the validity of a given formula, one searches for a set of complementary connections that spans the corresponding matrix. However, the existence of a spanning set for a matrix only ensures the validity of the corresponding formula in classical logic. For a characterization of validity in non-classical logics, additional requirements are necessary.

In particular, one must respect the peculiarities of non-classical context management. For various modal logics and intuitionistic logic, Wallen has a proposed a technique that is based on prefixes and string substitutions (Wallen, 1990). Prefixes represent context management in these logics in a very compact way, which makes them appealing for automating proof search. Efficient string unification algorithms for prefixes exist (Otten and Kreitz, 1996), proof search procedures based on Wallen's representations have been developed (Kreitz and Otten, 1999), and experiments with implementations of these procedures have demonstrated the advantages of the approach (Schmitt et al., 2001).

Formally, prefixes are strings of so-called *special positions*. A special position is either *constant* or *variable*, where variables can be substituted by (strings of) constants or other variables. In the construction of a matrix from a formula tree, special positions are inserted into the formula tree as additional nodes with the following intuition: *Inserting a constant special position encodes that the rule for reducing one of the adjacent nodes has a non-classical context management. Inserting a variable special position encodes that an adjacent position is a permissible context formula of such a rule application.* In the modal logic S4, for instance, the insertion of special positions is motivated by the non-classical context management of the $\Box$-rule and, hence, special positions are inserted next to nodes with connective $\Box$ or $\Diamond$. The *prefix* of a given position is the sequence of all special positions from the root of the modified formula tree to that position. If prefixes of different formulas can be unified then these formulas may move into the same branch of a corresponding sequent calculus proof.

Let us illustrate how the non-classical context management of $\mathcal{MELL}$ can be encoded by special positions. To simplify this illustration, we employ the unary calculus Σ_1 although our definition of matrices in Section 3.2 will be based on Σ_3'. In Σ_1, only the rules β and π have a non-classical context management. Classical variants of these rules would look as follows:

$$\frac{\Upsilon_1, \Upsilon_2, succ_1(\beta) \qquad \Upsilon_1, \Upsilon_2, succ_2(\beta)}{\Upsilon_1, \Upsilon_2, \beta} \qquad\qquad \frac{\boldsymbol{\nu}, \Upsilon, succ_1(\pi)}{\boldsymbol{\nu}, \Upsilon, \pi}$$

However, classical context management is *not* sound for these rules.[4]

In order to reflect the context management of the β-rule properly, *constant* positions from a set Ψ^M are inserted in between formulas of type β and their major sub-formulas. There are no restrictions on the type of context formulas for the β-rule. Hence, *variable* positions from a set Φ^M are inserted before *all* sub-formulas. During matrix-based proof search, the substitution of a variable by a string containing a specific constant shall encode that in corresponding sequent proofs the respective sub-formula moves into a particular branch. Hence, all inserted constants and variables have to differ from each other.

The non-classical context management for the π-rule differs from that for the β-rule, since all context formulas must have the particular type ν. We distinguish special positions that capture context management for the β-rule (the M in Ψ^M, Φ^M stands for *multiplicative*) from those that capture context management for the π-rule. To reflect context management for the π-rule, constant positions from a set Ψ^E (E stands for *exponential*) are inserted between formulas of type π and their major sub-formulas. Since ν-formulas may be in the context of an application of π, variable positions from a set Φ^E are inserted in between formulas of type ν and their major sub-formulas, which makes the insertion of a Φ^M-position at that position superfluous.

Since ν-formulas may be in the context of the π-rule as well as of the β-rule, arbitrary special positions (from Ψ^M, Φ^M, Ψ^E, Φ^E) may be substituted for variable positions in Φ^E. However, only multiplicative special positions (from Ψ^M, Φ^M) may be substituted for Φ^M-positions.

The complexity of string unification depends on the length of the strings involved. For an efficient matrix-based proof search it is desirable to avoid the insertion of special positions if this is possible without affecting soundness. For this reason, we base our formalization of matrices in Section 3.2 on triadic sequents rather than on unary ones and exploit the focusing principle to avoid inserting unnecessary special positions. By employing the focusing principle, we only need to insert special positions between certain *layers* of formulas rather than for every connective.

3.2. POSITION TREES AND MATRICES

Matrices are the fundamental syntactic construct in matrix based proof search. They are the objects for which proofs are constructed. Clearly, a close relation between matrices and formulas is necessary to ensure that a matrix is provable if and only if the corresponding formula is valid.

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[4] In the β-rule context formulas may only move into one of the two premises and all context formulas in the π-rule must be of type ν.

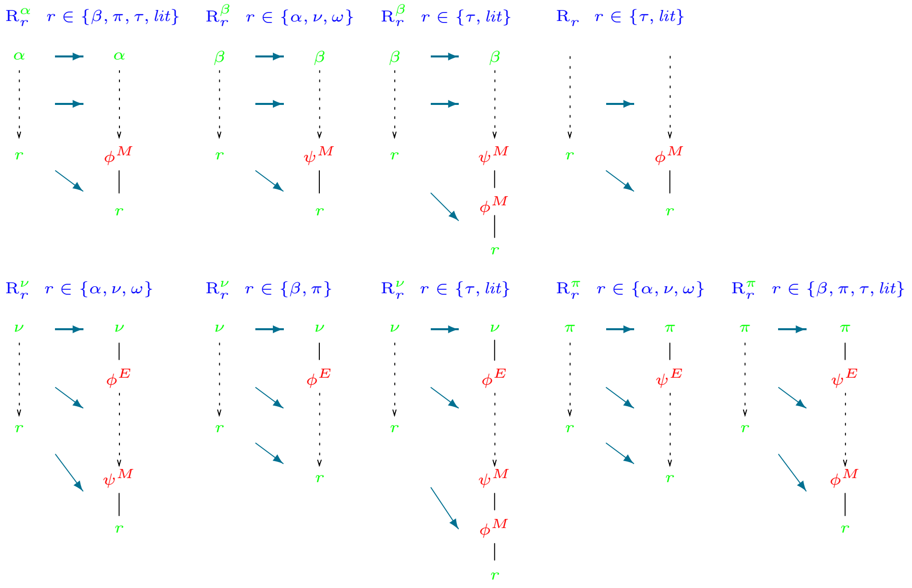


Figure 4. Rules for inserting special positions into basic position trees

The *matrix for a formula* is derived from the formula tree. First, each node of the formula tree is replaced by a basic position that serves as a reference to the original contents of the node. Then a *basic position tree* is constructed by inserting additional nodes, the *special positions*, which will later give rise to *prefixes*. Finally, a *position tree*, the matrix, is constructed by taking multiplicities into account.

Let V_b be an arbitrary set of *basic positions* such that for each $bp \in V_b$ the functions *lab*, *Ptype*, and *pol* are defined, $lab(bp)$ is a $\mathcal{MELL}$-formula, $Ptype(bp) \in \{lit, o, \tau, \omega, \alpha, \beta, \nu, \pi, \phi^M, \psi^M, \phi^E, \psi^E\}$, and $pol(bp) \in \{+, -\}$. A basic position with type *lit* is called *atomic*. In addition to the types known from uniform notation for $\mathcal{MELL}$-formulas in Section 2.1, basic positions may have the *special types* ϕ^M, ψ^M, ϕ^E, and ψ^E. Such basic positions are called *special basic positions*.

We define additional functions on basic positions. *con* returns the main connective of $lab(bp)$ and $sform(bp) = \langle lab(bp), pol(bp) \rangle$.

DEFINITION 5. Let φ be a $\mathcal{MELL}$-formula and T_φ be the tree that results from replacing each node in the formula tree of φ by a basic position for which *lab* yields the label, *pol* the polarity, and *Ptype* the type of that node. The *basic position tree* for φ results from applying the rewrite rules in Figure 4 to T_φ until none of them is applicable.

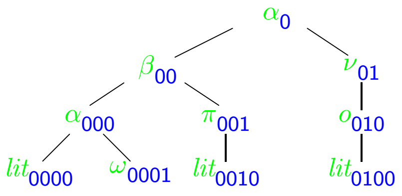
Each rewrite rule $R_{t_2}^{t_1}$ in Figure 4 inserts special positions wherever a sub-formula of type t_1 has a sub-formula of type t_2. It is applicable if the left hand pattern matches a subtree, where the dotted lines in the patterns match arbitrary subtrees that contain only nodes of type o. The rules R_{τ} and R_{lit} are a special case: they can only be applied if there are just positions of type o between the root and the leaf. If its left hand pattern matches a subtree, the rule rewrites the subtree according to the pattern on its right hand side.

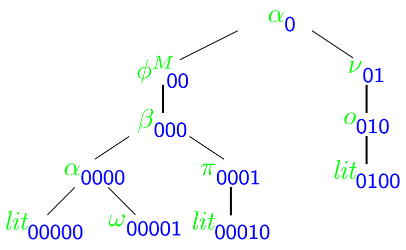
The rewrite rules insert special basic positions and modify edges, but do not remove nodes. Applying one rewrite rule to a tree cannot influence the applicability of any other rule. A close look at the rules in Figure 4 shows that the rewrite system is locally confluent. Since formula trees are finite, the notion of a basic position tree is well defined.

Given a basic position tree $T_b = (V, E)$ we extend the definition of some functions and relations to the inserted positions. $\mathit{Ptype}(\phi^M){=}\phi^M$, $\mathit{Ptype}(\psi^M){=}\psi^M$, $\mathit{Ptype}(\phi^E){=}\phi^E$, and $\mathit{Ptype}(\psi^E){=}\psi^E$. The application of lab and pol on an inserted special position bp shall yield the label and polarity of the successor node, if $\mathit{Ptype}(\mathit{bp}){\neq}\phi^E$, and of the predecessor node in the tree, if $\mathit{Ptype}(\mathit{bp}){=}\phi^E$. succ_1 and succ_2 are re-defined such that they yield the basic position that is respectively the left and right successor in T_b. Accordingly the orderings $\prec$ and $\ll$ are re-defined such that they reflect the tree structure of T_b.

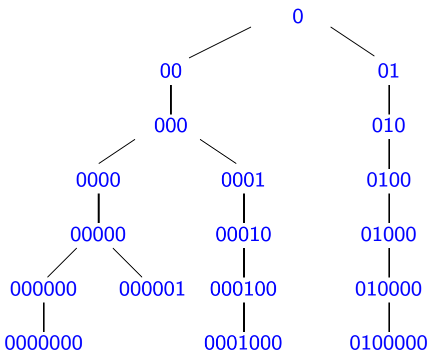
We denote basic positions in a basic position tree T_b by strings over $\{0, 1\}$. The root of T_b is denoted by 0. If a string s denotes a basic position then $\mathit{succ}_1(s)$ and $\mathit{succ}_2(s)$ are respectively denoted by $s0$ and $s1$, if they exist.

EXAMPLE 6. We explain the application of a rewrite rule at the example of the rule $\mathrm{R}^{\alpha}_{\beta}$. The formula tree for $F_1 = ((A \parr \bot) \otimes !A) \parr ?(A^{\perp})$, given by positions and types is displayed below on the left.





β_{00} is a successor of α_0 with no nodes in between. Thus, the subtree consisting of β_{00}, α_0, and the edge which links these nodes matches the left hand side pattern of the rule $\mathrm{R}^{\alpha}_{\beta}$. The tree is rewritten to the tree depicted on the right. Further applications of the rewrite rules $\mathrm{R}^{\beta}_{\alpha}$, $\mathrm{R}^{\alpha}_{lit}$, R^{π}_{lit}, and R^{ν}_{lit} result in the following basic position tree for F_1



bp	*Ptype*(*bp*)	*con*(*bp*)
0	α	⅋
00	ϕ^M	$\otimes$
000	β	$\otimes$
0000	ψ^M	⅋
00000	α	⅋
000000	ϕ^M	A
0000000	*lit*	A
000001	ω	$\bot$
0001	π	!
00010	ψ^E	!
000100	ϕ^M	A
0001000	*lit*	A
01	ν	?
010	ϕ^E	?
0100	o	$\bot$
01000	ψ^M	A
010000	ϕ^M	A
0100000	*lit*	A

The principal type and connectives for the basic positions are depicted in the table on the right. Note that position 00010 has connective ! and not A. ◇

By construction there is a one-to-one correspondence between non-special basic positions in a basic position tree for φ and the occurrences of sub-formulas in the formula tree of φ. To achieve a one-to-one correspondence between non-special positions and occurrences of sub-formulas of φ in a derivation we define *position trees* using multiplicities. A position tree is constructed from a basic position tree by creating multiple instances of the subtrees of ν-nodes. Formally it is defined by induction over the structure of the basic positions.

DEFINITION 7. Let $T_b = (V_b, E_b)$ be the basic position tree for a $\mathcal{MELL}$-formula φ, and μ be a multiplicity function on basic positions.

The *positions* $p{\in}V$ for φ and μ and the associated basic positions $bp(p){\in}V_b$ are defined recursively as follows.

$0 \in V$ and $bp(0) = 0$

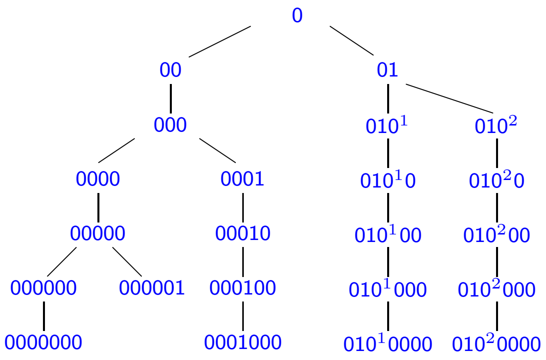
Let $p \in V$ and $bp(p) = p_b$.
If $Ptype(p_b) \in \{\alpha, \beta\}$, then $p0, p1 \in V$ and $bp(p0) = p_b0$, $bp(p1) = p_b1$.
If $Ptype(p_b) \in \{o, \pi, \phi^M, \psi^M, \phi^E, \psi^E\}$, then $p0 \in V$ and $bp(p0) = p_b0$.
If $Ptype(p_b) = \nu$, then $p0^i \in V$ and $bp(p0^i) = p_b0$ for $0{<}i{\leq}\mu(p_b)$.

The *position tree* for φ and μ is a tree $T = (V, E)$, where V is the set of positions for φ and μ and E is the set of edges which contains an edge from each position $p \in V$ to all positions $p0$, $p1$, and $p0^i$ in V.

For each position p the functions *lab*, *pol*, *Ptype*, *con*, and *sform* are defined to have the same values as for the associated basic positions. Positions of type *lit* are called *atomic* and positions of special type (ϕ^M,

ψ^M, ϕ^E, ψ^E) are called *special positions*. The functions $succ_1$, $succ_2$, and $succ_1^i$ yield the appropriate successors of a position. For notational convenience we denote positions by strings over $\{0,1,0^i\}$, $i \in \mathbb{N}$.

EXAMPLE 8. The position tree for $F_1 = ((A \parr \bot) \otimes !A) \parr ?(A^\bot)$ and the multiplicity μ with $\mu(01) = 2$ is depicted below. The principal type and connectives for the positions are depicted in the table on the right.



p	$Ptype(p)$	$con(p)$
0	α	$\parr$
00	ϕ^M	$\otimes$
000	β	$\otimes$
0000	ψ^M	$\parr$
00000	α	$\parr$
000000	ϕ^M	A
0000000	lit	A
000001	ω	$\bot$
0001	π	$!$
00010	ψ^E	$!$
000100	ϕ^M	A
0001000	lit	A
01	ν	$?$
010^i	ϕ^E	$?$
010^i0	o	$\bot$
010^i00	ψ^M	A
010^i000	ϕ^M	A
010^i0000	lit	A

In the *connection method*, the formula tree of a formula φ is usually represented in a two-dimensional fashion in order to illustrate the proof search techniques. A position tree is therefore also called a *matrix*.

DEFINITION 9. The *matrix for a $\mathcal{MELL}$-formula φ and a multiplicity μ* is the position tree for φ and μ.

4. A Sequent Calculus for $\mathcal{MELL}$-Matrices

As another intermediate step towards the matrix characterization of logical validity in $\mathcal{MELL}$ we will define a sequent calculus Σ_{pos} for matrices. This calculus operates on the *positions* of a formula tree instead of on the sub-formulas themselves, which eliminates a lot of redundancy from the proof. Though the use of positions might appear subtle at first they simplify the proof of the characterization theorem in the subsequent section.

The notion of a matrix can be extended straightforwardly to sequents. Each zone in a triadic sequent will be represented by a multi-set of position trees. Again, it will be necessary to insert special positions to separate layers of positions from each other.

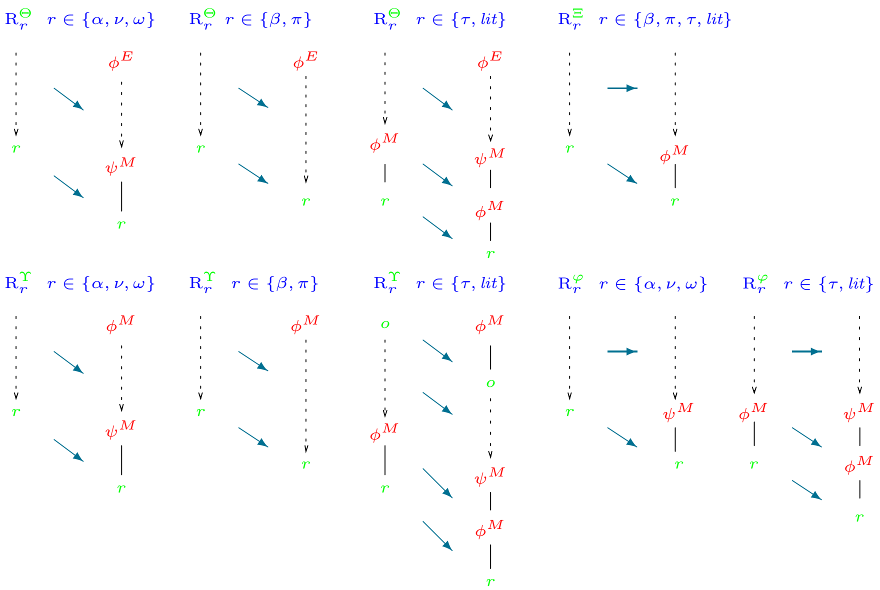


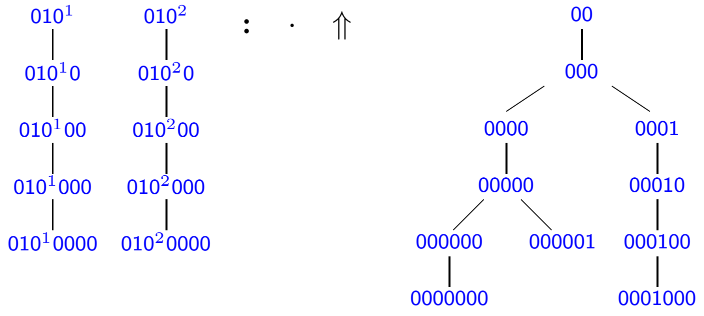
Figure 5. Rules for inserting special positions into basic position forests

DEFINITION 10. Let $S = \ \Theta : \Upsilon \Uparrow \Xi$ (or $\Theta : \Upsilon \Downarrow \varphi$) be a triadic sequent. Let $\mathcal{T}_u$, $\mathcal{T}_b$, and $\mathcal{T}_f$ be the multi-sets of position trees for the formulas in the unbounded, bounded, and focused zone of S, to which the rewrite rules in Figure 5 have been applied until none of them is applicable anymore. Then the triple $(\mathcal{T}_u, \mathcal{T}_b, \mathcal{T}_f)$ together with the mode $\Uparrow$ (or $\Downarrow$) is the *matrix M for the sequent S*.

The exponent of the rewrite rules in Figure 5 defines the zone in which the rules can be applied. Rules $\mathrm{R}^{\Theta}_{?}$, $\mathrm{R}^{\Upsilon}_{r}$, R^{Ξ}_{r}, and R^{φ}_{r} can be applied to trees in the unbounded zone, bounded zone, focused zone (mode $\Uparrow$), and focused zone (mode $\Downarrow$), respectively. Rules $\mathrm{R}^{\Upsilon}_{\tau}$ and $\mathrm{R}^{\Upsilon}_{lit}$ require a node of type o at the root of the pattern, while there are no such restrictions for the other rules. Again, the rewrite system is locally confluent and the notion of matrix for a sequent is well defined.

According to the rewrite rules, position trees in $\mathcal{T}_u$ ($\mathcal{T}_b$) always have a root of type ϕ^E (ϕ^M). Trees in the focused zone have a root with type from $\{\alpha, \nu, o, \omega, \phi^M\}$ for mode $\Uparrow$ and of type $\{\beta, \pi, o, \psi^M, \psi^E\}$ for mode $\Downarrow$. We will frequently indicate position trees just by their root. This allows us to denote matrices like triadic sequents, e.g., a matrix with mode $\Uparrow$ and triple (Φ^E, Φ^M, Ξ) is denoted by $\Phi^E : \Phi^M \Uparrow \Xi$.

EXAMPLE 11. The matrix for $\langle A^{\perp}, +\rangle, \langle A^{\perp}, +\rangle : \cdot \Uparrow \langle (A \parr \perp) \otimes !A, +\rangle$, the triadic sequent in the Σ'_3-proof of F_1 (see Example 3) is depicted below. Note that this matrix consists entirely out of subtrees of the position tree for F_1.



◇

From the perspective of matrices, a sequent proof only decomposes the position tree of a formula into subtrees and moves them into the various zones. The sequent calculus Σ_{pos}, presented in Figure 6, takes this observation into account and operates directly on matrices As always, derivations of matrices are defined with respect to a fixed multiplicity μ. A signed formula φ is *derivable in Σ_{pos}* if the matrix for $\cdot : \cdot \Uparrow \varphi$ is derivable for some multiplicity.

A close look at the calculus reveals the motivation for the insertion of special positions. The rules *focus*$_1$, *focus*$_2$, ϕ^M, ψ^M, and ψ^E allow position trees to be moved between the different zones of sequent and modes to be changed. A special position encodes the potential to apply one of these rules to the sub-formula where it has been inserted.

When a triadic sequent is transformed into a matrix, the rewrite rules R_r^{Θ} guarantee that each tree in the unbounded zone has a root of type ϕ^E and the rewrite rules R_r^{Υ} guarantee that each tree in the bounded zone has a root of type ϕ^M. The insertion of a node of type ϕ^E by a rewrite rule R_r^{ν} expresses that a subtree may move into the unbounded zone during a derivation. Similarly, the insertion of a ϕ^M-node expresses that a subtree may move into the bounded zone.

Most Σ_{pos}-rules directly correspond to a rule of the triadic calculus in Figure 3. The rules corresponding to *axiom*, τ, $o\Downarrow$, $o\Uparrow$, ω, α, β, ν, *focus*$_1$, and *focus*$_2$ have the same name as in the triadic calculus. The rules corresponding to *defocus* and *switch* are ϕ^M and ψ^M. However, the π rule of Σ_{pos} does not switch the mode of the sequent as in the triadic calculus. The mode can only be switched by the rules ψ^M and ψ^E. A rewrite rule R_r^{π} guarantees that after a node of type π always a node of type ψ^E is inserted. Thus, the π rule together with the ψ^E rule represent the π rule of Σ_3'. The reduction of the π type formula and the switching of the mode are split into separate rule applications. In Σ_{pos}, only the ψ^E rule enforces an empty bounded context. For technical reasons, the ν rule has not been split into two rules, as uniformity might suggest. Therefore, there is no ϕ^E rule in Σ_{pos} and the movement between zones is done by an application of the ν rule.

IDENTITY

$$\frac{}{\Phi^E : \phi_1, \phi_2 \Uparrow \cdot}\ \mathit{axiom}^{*}$$

NEGATION

$$\frac{\Phi^E : \Phi^M \Downarrow \mathit{succ}_1(o)}{\Phi^E : \Phi^M \Downarrow o}\ o\!\Downarrow \qquad \frac{\Phi^E : \Phi^M \Uparrow \Xi, \mathit{succ}_1(o)}{\Phi^E : \Phi^M \Uparrow \Xi, o}\ o\!\Uparrow$$

MULTIPLICATIVE FRAGMENT

$$\frac{}{\Phi^E : \phi_1 \Uparrow \cdot}\ \tau^{**} \qquad \frac{\Phi^E : \Phi^M \Uparrow \Xi}{\Phi^E : \Phi^M \Uparrow \Xi, \omega}\ \omega$$

$$\frac{\Phi^E : \Phi^M \Uparrow \Xi, \mathit{succ}_1(\alpha), \mathit{succ}_2(\alpha)}{\Phi^E : \Phi^M \Uparrow \Xi, \alpha}\ \alpha \qquad \frac{\Phi_1^E : \Phi_1^M \Downarrow \mathit{succ}_1(\beta) \quad \Phi_2^E : \Phi_2^M \Downarrow \mathit{succ}_2(\beta)}{\Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta}\ \beta$$

EXPONENTIAL FRAGMENT

$$\frac{\Phi^E : \Phi^M \Downarrow \mathit{succ}_1(\pi)}{\Phi^E : \Phi^M \Downarrow \pi}\ \pi \qquad \frac{\Phi^E, \mathit{succ}_1^1(\nu), \ldots, \mathit{succ}_1^{\mu(\nu)}(\nu) : \Phi^M \Uparrow \Xi}{\Phi^E : \Phi^M \Uparrow \Xi, \nu}\ \nu$$

SPECIAL POSITIONS

$$\frac{\Phi^E : \Phi^M \Downarrow \mathit{succ}_1(\phi_1)}{\Phi^E, \phi_1 : \Phi^M \Uparrow \cdot}\ \mathit{focus}_1{}^{***} \qquad \frac{\Phi^E : \Phi^M \Downarrow \mathit{succ}_1(\phi_1)}{\Phi^E : \Phi^M, \phi_1 \Uparrow \cdot}\ \mathit{focus}_2{}^{***}$$

$$\frac{\Phi^E : \Phi^M, \phi_1^M \Uparrow \Xi}{\Phi^E : \Phi^M \Uparrow \Xi, \phi_1^M}\ \phi^M \qquad \frac{\Phi^E : \Phi^M \Uparrow \mathit{succ}_1(\psi_1^M)}{\Phi^E : \Phi^M \Downarrow \psi_1^M}\ \psi^M \qquad \frac{\Phi^E : \cdot \Uparrow \mathit{succ}_1(\psi_1^E)}{\Phi^E : \cdot \Downarrow \psi_1^E}\ \psi^E$$

$*$ In *axiom* $Ptype(p_1)=lit=Ptype(p_2)$, $lab(p_1)=lab(p_2)$, and $pol(p_1)\neq pol(p_2)$ must hold for $p_i=succ_i(\phi_i)$.
$**$ In the τ-rule $Ptype(succ_1(\phi_1))$ must be τ.
$***$ In the *focus*-rules $Ptype(succ_1(\phi_1))$ must not be in $\{lit, \tau\}$.

Figure 6. The sequent calculus Σ_{pos} for $\mathcal{MELL}$-matrices

THEOREM 12 (Soundness and Completeness of $\boldsymbol{\Sigma}_{pos}$). *A signed formula φ is derivable in Σ_{pos} if and only if it is derivable in Σ_3'.*

The proof of Theorem 12 can be found in Appendix B.

5. The Matrix Characterization

In this section we give a formal account of the matrix characterization of logical validity for $\mathcal{MELL}$. We aim at a characterization that is similar to the ones given in (Wallen, 1990) and includes additional concepts to account for resource sensitivity. We begin by introducing the necessary notation and by defining paths, connections, weakening maps, prefixes, and prefix substitutions, which are essential for describing complementarity in $\mathcal{MELL}$. We will then show that complementarity is necessary and sufficient for logical validity in $\mathcal{MELL}$.

5.1. BASIC CONCEPTS

We denote the set of positions in a matrix M by $Pos(M)$. The set of *axiom positions* $AxPos(M)$ contains all positions with principal type τ or *lit*. The set of *weakening positions* $WeakPos(M)$ contains all positions of type ω and all ν-positions with $\mu(\nu)$=0. The set of *leaf positions* is defined as $LeafPos(M) = AxPos(M) \cup WeakPos(M)$. $\boldsymbol{\beta}(M)$ is the set of all β-positions in $Pos(M)$. The sets $\Phi^M(M)$, $\Psi^M(M)$, $\Phi^E(M)$, and $\Psi^E(M)$ are defined accordingly. The set of *special positions* is defined as $SpecPos(M) = \Phi^M(M) \cup \Psi^M(M) \cup \Phi^E(M) \cup \Psi^E(M)$.

The following concepts are the basis for an efficient matrix-representation of sequent proofs, as they help capturing the essential parts of a whole class of proofs. We represent sequents by *paths* through a matrix.

DEFINITION 13. A *path* is a set of positions.

1. The set $Paths(T)$ of *paths through a position tree* T is defined recursively by

   $\{0\}$, the set containing the root of T, is a path through T.

   If $P \cup \{p\}$ is a path through T then the following are paths

   - $P \cup \{succ_1(p), succ_2(p)\}$ if $Ptype(p) = \alpha$
   - $P \cup \{succ_1(p)\}$ and $P \cup \{succ_2(p)\}$ if $Ptype(p) = \beta$
   - $P \cup \{succ_1(p)\}$ if $Ptype(p) \in \{o, \pi, \phi^M, \psi^M, \phi^E, \psi^E\}$
   - $P \cup \bigcup_{i \leq \mu(p)} \{succ_1^i(p)\}$ if $Ptype(p) = \nu$, and $\mu(\nu) > 0$

2. The *set of paths through a set of position trees* $\mathcal{F}_s$ is defined by
   - $Paths(\emptyset) = \{\emptyset\}$,
   - $Paths(\{T\} \cup \mathcal{F}_s') = \{P_1 \cup P_2 \mid P_1 \in Paths(T), P_2 \in Paths(\mathcal{F}_s')\}$.

3. The *set of paths through a matrix* is defined by
   - $Paths(\Phi^E : \Phi^M \Downarrow \varphi) = Paths(\Phi^E \cup \Phi^M \cup \{\varphi\})$
   - $Paths(\Phi^E : \Phi^M \Uparrow \Xi) = Paths(\Phi^E \cup \Phi^M \cup \Xi)$.

A *path of leaves through* M is a subset of $LeafPos(M)$. $LPaths(M)$ denotes the set of all paths of leaves through M. Since leaf positions are not decomposed in the definition of paths, a path of leaves contains only irreducible positions.

A connection in a matrix represents a potential leaf in a sequent proof. While in classical, modal, and intuitionistic logic, connections require two literals with the same label but different polarities, the τ-rule in $\mathcal{MELL}$ also allows unary connections.

DEFINITION 14. A *connection* in a matrix M is a subset C of $AxPos(M)$ where either $C{=}\{p_1, p_2\}$, $Ptype(p_i){=}lit$, $lab(p_1){=}lab(p_2)$, and $pol(p_1){\neq}pol(p_2)$, or $C{=}\{p_1\}$ and $Ptype(p_1){=}\tau$.

A connection *C is on a path P* if $C{\subseteq}P$.

The restricted application of weakening in $\mathcal{MELL}$ also requires a novel concept of weakening maps, which contain all the positions that are explicitly weakened by applying the rules ω and ν or implicitly weakened when applying *axiom* and τ.

DEFINITION 15. A *weakening map* for M is a subset of $\Phi^E(M) \cup \mathit{WeakPos}(M)$.

Prefixes and prefix substitutions help to determine whether a proof branch described by a connection can actually be generated by a series of sequent rule applications. Since the only crucial rules are those that affect the context of a formula, we define a prefix of a position p to be the string of special positions between the root and p. Unifying the prefixes of connected positions then helps to determine the order in which non-permutable rules have to be applied as well as how to partition the context. Thus prefixes form the basis for a search process in which the necessary choices are made in a goal directed way.

DEFINITION 16. The *prefix* $\mathit{pre}_M(p)$ of a position $p \in \mathit{Pos}(M)$ is defined by

$$\mathit{pre}_M(0) = \begin{cases} r_0 & \text{if } r = \mathit{Ptype}(0) \in \{\phi^M, \psi^M, \phi^E, \psi^E\} \\ \varepsilon & \text{otherwise} \end{cases}$$

$$\mathit{pre}_M(p'i) = \begin{cases} \mathit{pre}_M(p')r_{p'i} & \text{if } r = \mathit{Ptype}(p'i) \in \{\phi^M, \psi^M, \phi^E, \psi^E\} \\ \mathit{pre}_M(p') & \text{otherwise} \end{cases}$$

If $p_1 \ll p_2$, i.e. if p_1 is a predecessor of p_2 in the position tree, then $\mathit{pre}_M(p_1)$ is an *initial substring* of $\mathit{pre}_M(p_2)$. We denote this by $\mathit{pre}_M(p_1) \angle \mathit{pre}_M(p_2)$.

DEFINITION 17. A *multiplicative prefix substitution* is a mapping $\sigma_M \colon \Phi^M \to (\Phi^M \cup \Psi^M)^*$. An *exponential prefix substitution* is a mapping $\sigma_E \colon \Phi^E \to (\Phi^M \cup \Psi^M \cup \Phi^E \cup \Psi^E)^*$. A *multiplicative exponential prefix substitution* is a mapping $\sigma \colon (\Phi^M \cup \Phi^E) \to (\Phi^M \cup \Psi^M \cup \Phi^E \cup \Psi^E)^*$ that maps elements from Φ^M to strings over $(\Phi^M \cup \Psi^M)^*$.

Substitutions are extended homomorphically to strings.

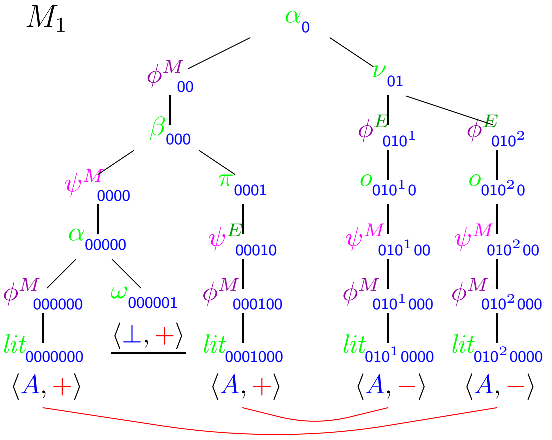
A prefix substitution substitutes elements from Φ^M and Φ^E by strings of special positions. Therefore, elements from Φ^M and Φ^E are called *variable special positions* while elements from Ψ^M and Ψ^E are called *constant special positions*. Note that according to the rules for inserting special positions in Figures 4 and 5 constant and variable special positions alternate within a prefix. A substitution for which all images contain only constant positions, is called a *grounded prefix substitution*. Figure 7 summarizes the relations between concepts defined in this subsection and sequent proofs.

basic concept	corresponding concept in sequent proof
matrix	formula or sequent
path	subsumes sequent that may occur in the proof
connection	application of *axiom* or τ
position in weakening map	application of weakening (ω, w)
constant special position	necessity to apply rule that affects context
variable special position	possible context formula in a rule application
substitution of a variable	application of a rule that affects the context

Figure 7. Relation of basic concepts to sequent proofs

EXAMPLE 18. The matrix M_1 for $F_1 = ((A \parr \bot) \otimes !A) \parr ?(A^\perp)$ and the multiplicity μ with $\mu(01) = 2$ is depicted below (c.f. Example 8).

Positions are marked with their principal types. For leaf positions we also show their connectives and polarities, which we underline in the case of weakening positions. The prefixes of leaf positions are given in the tables to the right, where we mark variable special positions by an overbar. The paths of leaves through M_1 are depicted below the matrix.



p	$pre_M(p)$
0000000	$\overline{\phi^M_{00}}\,\psi^M_{0000}\,\overline{\phi^M_{000000}}$
000001	$\overline{\phi^M_{00}}\,\psi^M_{0000}$
0001000	$\overline{\phi^M_{00}}\,\psi^E_{00010}\,\overline{\phi^M_{000100}}$
$010^1 0000$	$\overline{\phi^E_{010^1}}\,\psi^M_{010^1 00}\,\overline{\phi^M_{010^1 000}}$
$010^2 0000$	$\overline{\phi^E_{010^2}}\,\psi^M_{010^2 00}\,\overline{\phi^M_{010^2 000}}$

$$\begin{aligned}
\sigma_1(\overline{\phi^M_{00}}) &= \varepsilon \\
\sigma_1(\overline{\phi^E_{010^1}}) &= \psi^E_{00010} \\
\sigma_1(\overline{\phi^E_{010^2}}) &= \psi^M_{0000} \\
\sigma_1(\overline{\phi^M_{000000}}) &= \psi^M_{010^2 00} \\
\sigma_1(\overline{\phi^M_{000100}}) &= \psi^M_{010^1 00} \\
\sigma_1(\overline{\phi^M_{010^1 000}}) &= \varepsilon \\
\sigma_1(\overline{\phi^M_{010^2 000}}) &= \varepsilon
\end{aligned}$$

$LPaths(M_1) = \{\ \{0000000,\ 000001,\ 010^1 0000,\ 010^2 0000\}, \{0001000,\ 010^1 0000,\ 010^2 0000\}\ \}$

In the matrix we indicate connections in the set $\mathcal{C}_1$ by curved lines between axiom positions and underline all positions in the weakening map. As weakening map for M_1 we select $\mathcal{W}_1 = \{000001\}$. The multiplicative exponential prefix substitution σ_1 that unifies the prefixes of the connected positions is given below the table of prefixes. ◇

5.2. Complementarity

Matrix characterizations for classical (Bibel, 1987) and non-classical (Wallen, 1990; Waaler, 2001) logics are based on a notion of *complementarity*. Essentially this means that every path through a matrix must contain a unifiable connection. This requirement also holds for linear logic but has to be refined by additional properties, which we shall specify now. In the following we always assume M to be a matrix, $\mathcal{C}$ to be a set of connections, $\mathcal{W}$ to be a weakening map for M, and σ to be a prefix substitution.

The *spanning property* is the most fundamental requirement. Each path of leaves must contain a connection. A set of connections $\mathcal{C}$ *spans* a matrix M iff for every path $P \in \mathit{LPaths}(M)$ there is a connection $C \in \mathcal{C}$ with $C \subseteq P$.

The *unifiability property* states that the prefixes of connected positions must be made identical and, due to the restricted application of weakening, that each position in a weakening map must be related to a connection. $\mathcal{C}, \mathcal{W}$ is *unifiable* if there exists a prefix substitution σ such that (1) for all $\{p_1, p_2\} \in \mathcal{C}\ \ \sigma(\mathit{pre}_M(p_1)) = \sigma(\mathit{pre}_M(p_2))$ and (2) for all $wp \in \mathcal{W}$ there is some $C = \{p, \ldots\} \in \mathcal{C}$ with $\sigma(\mathit{pre}_M(wp)) \underline{\angle} \sigma(\mathit{pre}_M(p))$. In this case σ is called a *unifier* for $\mathcal{C}, \mathcal{W}$.

A unifier σ of $\mathcal{C}, \mathcal{W}$ can always be modified to a unifier σ' such that $\sigma'(\mathit{pre}_M(\phi^E)) = \sigma'(\mathit{pre}_M(p))$ holds for all $\phi^E \in \mathcal{W}$ and some connection $\{p, \ldots\} \in \mathcal{C}$. A *grounded substitution* σ' can be constructed from a substitution σ by removing all variable positions from values. If σ is a unifier for $\mathcal{C}, \mathcal{W}$ then the σ' is a unifier as well.

The *linearity property* expresses that no resource is to be used twice, i.e. contraction is restricted. A resource cannot be connected more than once and cannot be connected at all if that part of the formula is weakened. $\mathcal{C}, \mathcal{W}$ is *linear* for M iff (1) any two connections $C_1 \neq C_2 \in \mathcal{C}$ are disjoint and (2) no ϕ^E in the prefix of a connected position p (i.e. $\{p, \ldots\} \in \mathcal{C}$) belongs to the set $\mathcal{W}$.

The *relevance property* requires each resource to be used at least once. A resource is used if it is connected or weakened. $\mathcal{C}, \mathcal{W}$ has the *relevance property* for M if for $p \in \mathit{LeafPos}(M)$ either (1) $p \in C$ for some $C \in \mathcal{C}$, (2) $p \in \mathcal{W}$, or (3) some ϕ^E in the prefix of p belongs to $\mathcal{W}$.

The *cardinality property* expresses that the number of branches in a sequent proof is adequate. It substitutes the *minimality property* in (Fronhöfer, 1996), which would require a complicated test for $\mathcal{MELL}$.

$\mathcal{C}, \mathcal{W}$ has the *cardinality property* for M if $|\mathcal{C}| = |\boldsymbol{\beta}(M)| - \sum_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)| + 1$. The right hand side of this equation determines the number of branches in a corresponding sequent proof by counting the number of positions of type β and then subtracting those that are weakened (note that ϕ^E denotes both a weakening position and the subtree beginning at that position) and, thus, are not reduced.

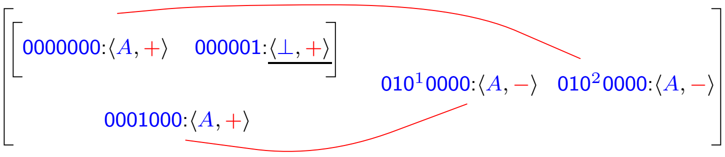
Each of the above requirements captures an essential aspect of a sequent proof. The spanning property expresses that all proof branches are covered by connections, while the unifiability of prefixes guarantees that connected positions can be moved into the same proof branch and that positions in $\mathcal{W}$ can be weakened in some branch. Linearity and relevance express the absence of implicit contraction and weakening, while cardinality expresses the absence of the rule of mingle, i.e. a proof can only branch when reducing β-type positions. Taken together, these properties ensure the existence of a sequent proof.

DEFINITION 19. (Complementarity)
A matrix M is *complementary* iff there are a set of connections $\mathcal{C}$, a weakening map $\mathcal{W}$, and a prefix substitution σ such that
$\mathcal{C}$ spans M,
σ is a unifier for $\mathcal{C}, \mathcal{W}$, and
$\mathcal{C}, \mathcal{W}$ is linear for M and has the relevance and cardinality properties.
We also say that *M is complementary for $\mathcal{C}$, $\mathcal{W}$ and σ*.

Since the notion of complementarity captures the essential aspects of Σ_{pos}-proofs while omitting the unimportant details, the search space is once more compactified. Problems like context splitting at the reduction of β-type positions simply do not occur, as this information can be determined by unifying the prefixes of connected positions.

EXAMPLE 20. Consider again the matrix M_1 from Example 18. We will show that M_1 is complementary for the corresponding set of connections, the weakening map, and the prefix substitution.

In a two-dimensional representation of the leaf positions of M_1 the leaf paths through M_1 correspond to horizontal lines through the positions. Connections are marked by curved lines. It is easy to see that each path contains a connection from $\mathcal{C}_1$, which means that $\mathcal{C}_1$ spans M_1.



Applying σ_1 to the prefixes of the leaf positions yields

$$\begin{array}{lcl}
\sigma_1(pre_M(\mathsf{0000000})) & = & \psi^M_{\mathsf{0000}}\ \psi^M_{\mathsf{010^2 00}} \\
\sigma_1(pre_M(\mathsf{010^2 0000})) & = & \psi^M_{\mathsf{0000}}\ \psi^M_{\mathsf{010^2 00}} \\
\sigma_1(pre_M(\mathsf{0001000})) & = & \psi^E_{\mathsf{00010}}\ \psi^M_{\mathsf{010^1 00}} \\
\sigma_1(pre_M(\mathsf{010^1 0000})) & = & \psi^E_{\mathsf{00010}}\ \psi^M_{\mathsf{010^1 00}} \\
\sigma_1(pre_M(\mathsf{000001})) & = & \psi^M_{\mathsf{0000}}
\end{array}$$

Thus σ_1 unifies the prefixes of the connected axiom positions. The prefix of the weakening position 000001 is mapped into an initial substring of $\sigma_1(pre_M(0000000))$.

$\mathcal{C}_1, \mathcal{W}_1$ is linear, because the connections are disjoint and 000001, the only position in $\mathcal{W}_1$, does not occur in any prefix. $\mathcal{C}_1, \mathcal{W}_1$ is relevant because all leaf positions belong either to a connection or are elements of the weakening map.

There are two connections in $\mathcal{C}_1$ and one β-type position in M_1. Hence $|\mathcal{C}_1| = 2 = |\boldsymbol{\beta}(M_1)| - \sum_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)| + 1$, as $\mathcal{W}_1$ does not contain any ϕ^E-positions. This means that $\mathcal{C}_1, \mathcal{W}_1$ has the cardinality property.

Thus all requirements for the complementarity of M_1 are fulfilled. ◇

Complementarity is the foundation for matrix based proof techniques. It yields a compact representation of the search space, which can be exploited by proof search methods in the same way as for classical, intuitionistic, and modal logics (Kreitz and Otten, 1999). These methods have already been extended to $\mathcal{MLL}$, as shown in (Kreitz et al., 1997). Along the same lines an extension to $\mathcal{MELL}$ is possible, since our definition of complementarity is necessary and sufficient for proving the logical validity of $\mathcal{MELL}$-formulas.

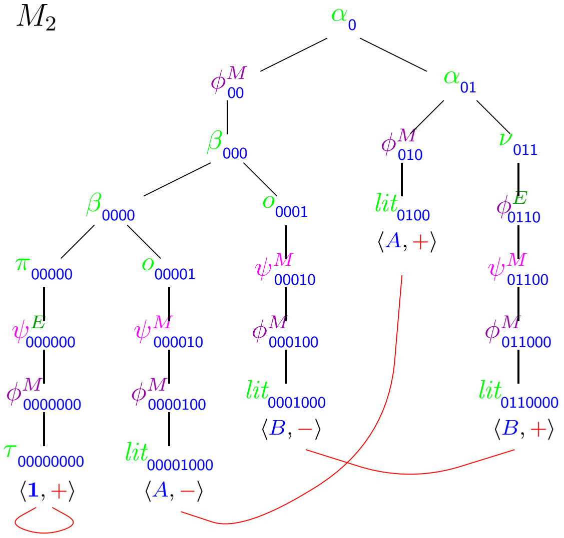
THEOREM 21 (Characterization Theorem).
A formula φ is valid in $\mathcal{MELL}$ if and only if the corresponding matrix is complementary for some multiplicity.

A proof of this theorem will be given in Section 6.

The following example illustrates some of the other principles of our matrix characterization such as the use of unary connections, and the significant optimizations that are gained by using the focusing principle in our matrix characterization.

EXAMPLE 22. The matrix M_2 for $F_2 = ((!\mathbf{1} \otimes A^{\perp}) \otimes B^{\perp}) \parr (A \parr ?B)$ is depicted below.

M_2



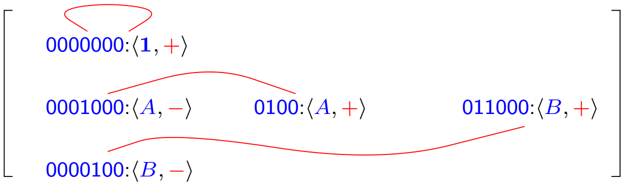
p	$pre_M(p)$
00000000	$\overline{\phi^M_{00}}\,\psi^E_{000000}\,\overline{\phi^M_{0000000}}$
00001000	$\overline{\phi^M_{00}}\,\psi^M_{000010}\,\overline{\phi^M_{0000100}}$
0001000	$\overline{\phi^M_{00}}\,\psi^M_{00010}\,\overline{\phi^M_{000100}}$
0100	$\overline{\phi^M_{010}}$
0110000	$\overline{\phi^E_{0110}}\,\psi^M_{01100}\,\overline{\phi^M_{011000}}$

$$\begin{aligned}
\sigma_2(\overline{\phi^M_{00}}) &= \varepsilon \\
\sigma_2(\overline{\phi^M_{0000000}}) &= \varepsilon \\
\sigma_2(\overline{\phi^M_{0000100}}) &= \varepsilon \\
\sigma_2(\overline{\phi^M_{000100}}) &= \psi^M_{01100} \\
\sigma_2(\overline{\phi^M_{010}}) &= \psi^M_{000010} \\
\sigma_2(\overline{\phi^E_{0110}}) &= \psi^M_{00010} \\
\sigma_2(\overline{\phi^M_{011000}}) &= \varepsilon
\end{aligned}$$

$LPaths(M_2) = \{\ \{00000000,\ 0100,\ 0110000\},\ \{00001000,\ 0100,\ 0110000\},$
$\{0001000,\ 0100,\ 0110000\}\ \}$

Note that no special positions have been inserted between α_0, α_{01}, and ν_{011}. Similarly, no special positions have been inserted between β_{000}, β_{0000}, and π_{00000}. This is a consequence of using Andreoli's focusing principle in the construction of our matrix characterization. Without employing this principle, there would be two special positions (one constant, one variable) between every two of these adjacent positions and the largest prefix of a leaf position would consist of seven special positions instead of three.

As before, the set of connections $\mathcal{C}_2$ is indicated by curved lines between axiom positions (note the unary connection). The weakening map $\mathcal{W}_2$ is empty. A multiplicative exponential substitution σ_2 that unifies the prefixes of the connected positions is given below the table of prefixes. The paths of leaves through M_2 are depicted below the matrix. The following two-dimensional representation of M_2 shows that $\mathcal{C}_2$ spans M_2.



σ_2 is a unifier for $\mathcal{C}_2, \mathcal{W}_2$ since it unifies the prefixes of the connected atoms and $\mathcal{W}_2$ is empty. $\mathcal{C}_2, \mathcal{W}_2$ is linear, because the connections are disjoint and $\mathcal{W}_2$ is empty. $\mathcal{C}_2, \mathcal{W}_2$ is relevant because all leaf positions belong to a connection. $\mathcal{C}_2, \mathcal{W}_2$ has the cardinality property, since there are three connections in $\mathcal{C}_2$ and two β-type positions in M_2, while $\mathcal{W}_2$ is empty. Thus M_2 is complementary and F_2 is valid in $\mathcal{MELL}$. ◇

5.3. DISCUSSION

Our matrix characterization of $\mathcal{MELL}$ introduces some novel concepts compared to the matrix characterizations for other non-classical logics.

Linearity is necessary to avoid that a single resource is used twice. This is already emphasized by the linear connection method (Bibel, 1986), where the spanning and linearity properties are the only conditions in the definition of complementarity. The resulting logic is resource sensitive, albeit, differs from linear logic in important aspects.

Weakening maps, relevance, and the cardinality property were necessary because of the restriction of weakening in linear logic. The cardinality condition is needed, because relevance condition is not quite sufficient. It reflects that the rule of mingling is not part of the calculus.

$$\frac{\Gamma_1 \quad \Gamma_2}{\Gamma_1, \Gamma_2}\ \mathit{mingle}$$

The peculiarity of context handling in $\mathcal{MELL}$ is encoded by the prefixes. Inserted constant positions reflect the requirements on the context in the π-rule and on the partition in the β-rule. The use of Andreoli's focusing principle is not strictly necessary, but it helps reducing the number of special positions that have to be inserted tremendously. The unification property on these prefixes guarantees a handling of the context that is proper for $\mathcal{MELL}$.

Since our matrix characterization for $\mathcal{MELL}$ compactifies the search space significantly, it would be desirable to extend it to full linear logic. This, however, will very likely require even more novel concepts that are not known at this point. In particular, adapting the concept of prefixes to the additives will be a challenging issue. The difficulty lies in having additive connectives in addition to the multiplicative connectives. Prefixes for the additive fragment in isolation have been proposed by Galmiche in (Galmiche, 2000, §6.2). The rules

$$\frac{\Gamma, F_1}{\Gamma, F_1 \oplus F_2}\ \oplus_1 \qquad \frac{\Gamma, F_2}{\Gamma, F_1 \oplus F_2}\ \oplus_2$$

interfere with the relevance and cardinality properties, since the subformulas in F_1 (F_2) must not contribute to the proof. Adapting the definition of weakening maps might be a way to overcome this problem. The rule for additive conjunction duplicates formulas in the context.

$$\frac{\Gamma, F_1 \quad \Gamma, F_2}{\Gamma, F_1 \& F_2}\,\&$$

This interferes with the linearity and cardinality conditions, since the formulas in Γ may and must be used twice in the overall proof. A change in the definition of multiplicities may be necessary. However, coming up with a usable solution for & appears to be difficult.

The rule for the additive constant $\top$ (there is no calculus rule for $\mathbf{0}$)

$$\frac{}{\Gamma, \top}\,\top$$

interferes with the relevance and cardinality conditions. As for $\oplus$ an adaption of the definition of weakening maps appears to be reasonable. Alternatively one could adapt the definition of connections accordingly.

6. Proving Soundness and Completeness

The matrix-characterization of logical validity for a $\mathcal{MELL}$ formula φ requires the matrix for φ to be complementary for some multiplicity μ (see Definition 19). By proving the equivalence of this characterization to the derivability in Σ_{pos} we will prove it sound and complete. For this purpose we introduce a few technical notions.

Unifiability requires each position wp in a weakening map $\mathcal{W}$ to be related to some connection $C \in \mathcal{C}$ such that $\sigma(pre_M(wp)) \angle \sigma(pre_M(p))$ holds for $p \in C$. Let *AssSet(C)* be the union of C and the set of positions in $\mathcal{W}$ that are related to C. In principle, a position in the weakening map can be associated to more than one connection. However, we define $AssSet(C)$ such that each position in the weakening map is associated to exactly one connection, i.e. $\mathcal{W} \subseteq \bigcup_{C\in\mathcal{C}} AssSet(C)$ and $\forall C_1, C_2 \in \mathcal{C}.\ C_1{\neq}C_2 \implies AssSet(C_1){\cap}AssSet(C_2){=}\emptyset$.

A matrix M for a sequent is a multi-set of position trees, or a *position forest*. Let T_1 and T_2 be position trees in M. T_1 and T_2 are connected if there is a *link* between the positions of T_1 and those of T_2, where two positions are linked if they belong to some $AssSet(C)$, $C \in \mathcal{C}$. Formally, we define a relation $AssRel$ by

$$\begin{aligned} &AssRel(T_1, T_2)\\ &\qquad \textit{iff}\ \ \exists C{\in}\mathcal{C}.\ \exists p_1{\in}Pos(T_1).\ \exists p_2{\in}Pos(T_2).\ p_1, p_2 \in AssSet(C)\ . \end{aligned}$$

Let $\sim$ be the reflexive transitive closure of $AssRel$. A *connected component in M* is a position forest in M that is an equivalence class of $\sim$.

We define a function $FCons$ that restricts a set of connections $\mathcal{C}$ to those connections whose elements are contained within a given position forest $\mathcal{F}$. Accordingly, we define a function $FWeak$ that restricts a weakening map $\mathcal{W}$ to the positions contained within $\mathcal{F}$. We define

$$\begin{array}{lcl} FCons(\mathcal{F},\mathcal{C}) & = & \{C \in \mathcal{C} \mid \forall p \in C.p \in Pos(\mathcal{F})\} \\ FWeak(\mathcal{F},\mathcal{W}) & = & \{p \in \mathcal{W} \mid p \in Pos(\mathcal{F})\}\ . \end{array}$$

The following two lemmata show how the prefix substitution ensures proper context management. They state that the rules ψ^E and β are applicable in the Σ_{pos}-proof that we will construct from a matrix proof.

LEMMA 23. *Let $\mathcal{C},\mathcal{W}$ be spanning, linear, and have the relevance and cardinality properties for the matrix M and σ be a unifier for $\mathcal{C},\mathcal{W}$. Then there is exactly one connected component in M.*

Proof. We prove a generalized version of the lemma where M is an arbitrary position forest instead of a matrix. We first show that

$$|FCons(\mathcal{F},\mathcal{C})| \geq |\boldsymbol{\beta}(\mathcal{F})| - \sum_{\phi^E \in FWeak(\mathcal{F},\mathcal{W})} |\boldsymbol{\beta}(\phi^E)| + 1$$

holds for any position forest M and any connected component $\mathcal{F}$ in M if $\mathcal{C},\mathcal{W}$ is linear and relevant for M and σ is a unifier for $\mathcal{C},\mathcal{W}$. We prove the proposition by induction over the number of β-positions in M.

If $|\boldsymbol{\beta}(M)| = 0$ then $\boldsymbol{\beta}(M) \setminus \bigcup_{\phi^E \in \mathcal{W}} \boldsymbol{\beta}(\phi^E) = \emptyset$ and the proposition follows from the relevance property, as there is at least one connection.

The induction step uses another induction over the number of positions in M that are not of type β, *lit*, τ, or ω and do not have a predecessor of type β.

- In the base case all roots in M are of type β, *lit*, τ, or ω. We select a connected component $\mathcal{F}$ in M with a root q of type β, whose removal from $\mathcal{F}$ results in two connected components $\mathcal{F}_1$ and $\mathcal{F}_2$.[5] Then $|FCons(\mathcal{F},\mathcal{C})|$

  $$\begin{aligned} &= |FCons(\mathcal{F}_1,\mathcal{C})| + |FCons(\mathcal{F}_2,\mathcal{C})| \\ &\geq \quad |\boldsymbol{\beta}(\mathcal{F}_1)| - \textstyle\sum_{\phi^E \in FWeak(\mathcal{F}_1,\mathcal{W})} |\boldsymbol{\beta}(\phi^E)| + 1 \\ &\quad + |\boldsymbol{\beta}(\mathcal{F}_2)| - \textstyle\sum_{\phi^E \in FWeak(\mathcal{F}_2,\mathcal{W})} |\boldsymbol{\beta}(\phi^E)| + 1 \\ &\geq |\boldsymbol{\beta}(\mathcal{F})| - \textstyle\sum_{\phi^E \in FWeak(\mathcal{F},\mathcal{W})} |\boldsymbol{\beta}(\phi^E)| + 1 \end{aligned}$$

  Note that $\mathcal{C},\mathcal{W}$ are linear and relevant for the position forest M', which results from M by replacing $\mathcal{F}$ with $\mathcal{F}_1$ and $\mathcal{F}_2$, and σ is a unifier. Thus the proposition holds for M', $\mathcal{F}_1$, and $\mathcal{F}_2$ according to the assumption of the outer induction.

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[5] A clear separation into two connected components is possible, because the existence of a unifier excludes the possibility of β-*circles* in M: there cannot be a sequence of disjoint position pairs $\langle(p_{i1}, p_{o1}), \ldots, (p_{in}, p_{on})\rangle$ such that the p_{ij} and p_{oj} are linked through some $AssSet(C)$, while p_{oj} and p_{ij+1} as well as p_{on} and p_{i1} have a β-position as most recent common predecessor.

- In the step case there is a root q in M that is not of type β, *lit*, τ, or ω. Removing q from M results in a matrix M', to which the (inner) induction assumption can be applied. q must be chosen with some care and the prefix substitution may have to be adapted. There are four subsequent choices:
  1. If q can be chosen to have a type in $\{\alpha, o, \pi, \nu\}$ then $\mathcal{C}, \mathcal{W}$ fulfill the precondition of the lemma for M'.
  2. If q can be chosen as an element of $\mathcal{W}$ of type ϕ^E, then M' results from M by removing the whole tree with root q. $\mathcal{C}, \mathcal{W} \setminus \{\phi^E\}$ fulfill the precondition of the lemma for M'.
  3. If q can be chosen to have type ψ^M or ψ^E then $\mathcal{C}, \mathcal{W}$ and σ' fulfill the precondition of the lemma for M' where σ' results from σ by removing q from all images and M' results from M by removing only q.
  4. Otherwise, q must be of type ϕ^M or ϕ^E and not be in $\mathcal{W}$. If there are several choices, we select q such that the length of $\sigma(q)$ is minimal. $\mathcal{C}, \mathcal{W}$ and σ' fulfill the precondition of the lemma for M' where σ' results from σ by removing $\sigma(q)$ from all images.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be the connected components in M. Since the connections, weakening maps, and β-type positions of the $\mathcal{F}_i$ are completely disjoint, we have

$$\begin{aligned} |\mathcal{C}| &= \sum_{i=1\ldots n} |FCons(\mathcal{F}_i, \mathcal{C})| \\ &\geq \sum_{i=1\ldots n} (\, |\boldsymbol{\beta}(\mathcal{F}_i)| - \sum_{\phi^E \in FWeak(\mathcal{F}_i, \mathcal{W})} |\boldsymbol{\beta}(\phi^E)| + 1\,) \\ &= |\boldsymbol{\beta}(M)| - \sum_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)| + n \end{aligned}$$

Because of the cardinality property, n must be ≤ 1 and because of the spanning property there must be at least one connected component. □

LEMMA 24. *Let M be the matrix for $\Phi^E : \Phi^M \Downarrow \psi_1^E$. Let $\mathcal{C}, \mathcal{W}$ have the spanning, linearity, relevance, and cardinality properties for M and σ be a grounded prefix substitution that unifies $\mathcal{C}, \mathcal{W}$. Then $\Phi^M = \emptyset$.*

Proof. We first prove a few auxiliary properties

1. Let $T \neq \psi_1^E$ be a tree in M with root q and p, p' be arbitrary positions in T. If $\psi_1^E \underline{\angle}\, \sigma(pre_M(p))$ then $\psi_1^E \underline{\angle}\, \sigma(q)$ and $\psi_1^E \underline{\angle}\, \sigma(pre_M(p'))$. According to Definition 10, q is a (variable) position. Since variable and constant positions alternate in a prefix, $pre_M(p)$ is either q or of the form $q.\psi_0.s$, where $\psi_0 \neq \psi_1^E$. Since $\sigma(pre_M(p))$ begins with ψ_1^E, q must be mapped to a string that begins with ψ_1^E. $\psi_1^E \underline{\angle}\, \sigma(pre_M(p'))$ holds because q is a substring of $pre_M(p')$.

2. $\psi_1^E \underline{\angle}\, \sigma(pre_M(p))$ holds for all $p \in LeafPos(M)$ with the property that $\phi^E \ll p$ does not hold for any $\phi^E \in \mathcal{W}$ which has only non-special positions as predecessors.

Let $Aux_1 = \{p{\in}LeafPos(M) \mid \neg(\psi_1^E \underline{\angle}\, \sigma(pre_M(p)))\}$ and $Aux_2 = \{p{\in}LeafPos(M) \mid \neg\exists\phi^E{\in}\mathcal{W}.\phi^E{\ll}p \wedge \forall p'{\ll}\, \phi^E.Ptype(p') \notin \{\phi^M, \psi^M, \phi^E, \psi^E\}\ \}$ and assume that $Aux_1 \cap Aux_2$ is not empty. By Lemma 23 M has only one connected component. Thus, there is a tree T with leaf positions in $Aux_1 \cap Aux_2$ that is linked to a tree T' whose leaf positions are not in Aux_1. Let $p{\in}Pos(T)$ and $p'{\in}Pos(T')$ such that $p, p'{\in}AssSet(C)$ and $\{p, p'\} \cap C \neq \emptyset$ for some $C{\in}\mathcal{C}$. T, T', p, and p' can be chosen such that all these conditions hold.
Because of unifiability $\psi_1^E \underline{\angle}\, \sigma(pre_M(p')) \underline{\angle}\, \sigma(pre_M(p))$ holds if $p{\in}C$. Otherwise $\sigma(pre_M(p)) \underline{\angle}\, \sigma(pre_M(p'))$ and $\psi_1^E \underline{\angle}\, \sigma(pre_M(p'))$ hold. Because of the way special positions are inserted before weakening positions (type ω or ν), $pre_M(p)$ must contain a ψ-position. Thus $\sigma(pre_M(p))$ cannot be empty, which means that it must begin with the same position as $\sigma(pre_M(p'))$ or that a $\phi^E \in \mathcal{W}$ exists with $\phi^E \ll p$ which has only non-special positions as predecessors. In all these cases we receive a contradiction with the choice of T.

Since the root q of a tree in Φ^M is a position of type ϕ^M, it can only be substituted by special positions of type ϕ^M or ψ^M. On the other hand $\psi_1^E \underline{\angle}\, \sigma(q)$ must hold. Thus Φ^M must be empty. □

THEOREM 25 (Soundness).
If a matrix M is complementary for a set $\mathcal{C}$ of connections, a weakening map $\mathcal{W}$, and a substitution σ then there is a Σ_{pos}-proof $\mathcal{P}$ for M.

Proof. We use Noetherian induction on the *weight* *wgt(M)* of the matrix M, which we define as follows.

$$\begin{aligned} wgt(\Phi^E : \Phi^M \Downarrow F) &= \textstyle\sum_{\phi\in(\Phi^E\cup\Phi^M)}(2 * |Pos(\phi)|) + 2 * |Pos(F)| + 1 \\ wgt(\Phi^E : \Phi^M \Uparrow \Xi) &= \textstyle\sum_{\phi\in(\Phi^E\cup\Phi^M)}(2 * |Pos(\phi)|) + \sum_{p\in\Xi}(2 * |Pos(p)| + 1)\,. \end{aligned}$$

Note that for all Σ_{pos}-rules the weight of the premise(s) is less than the weight of the conclusion, since *wgt* takes multiplicities into account.

There is no base case. In the induction step we assume that the claim holds for all matrices with lesser weight than M and perform a complete case analysis on the structure of M.

$M = \Phi^E : \Phi^M \Uparrow \cdot$: We distinguish two cases.
If $succ_1(\phi) \in LeafPos(M)$ holds for all $\phi{\in}\Phi^M{\cup}(\Phi^E \setminus \mathcal{W})$ then (because of relevance and cardinality) either $\Phi^M{=}\{\phi_1^M, \phi_2^M\}$, $\Phi^E{\subseteq}\mathcal{W}$, and *axiom* is applicable because of $\{succ_1(\phi_1^M), succ_1(\phi_2^M)\} \in \mathcal{C}$, or $\Phi^M{=}\{\phi_1^M\}$, $\Phi^E{\subseteq}\mathcal{W}$ and τ is applicable or $\{succ_1(\phi_1^M)\}{\in}\mathcal{C}$.

If there is a $\phi{\in}\Phi^M \cup (\Phi^E \setminus \mathcal{W})$ with $succ_1(\phi) \notin LeafPos(M)$ then $focus_1$ or $focus_2$ is applicable for some $\phi'{\in}\Phi^M \cup (\Phi^E \setminus \mathcal{W})$ with $\sigma(\phi'){=}\varepsilon$ and the induction hypothesis can be applied.

$M = \Phi^E : \Phi^M \Uparrow \Xi, \omega$, $M = \Phi^E : \Phi^M \Uparrow \Xi, o$, $M = \Phi^E : \Phi^M \Uparrow \Xi, \alpha$, $M = \Phi^E : \Phi^M \Uparrow \Xi, \phi_1^M$, $M = \Phi^E : \Phi^M \Uparrow \Xi, \nu$, $M = \Phi^E : \Phi^M \Downarrow o$, or $M = \Phi^E : \Phi^M \Downarrow \pi$:

In these cases we first use the rules ω, $o \Uparrow$, α, ϕ^M, ν, $o \Downarrow$, or π and then apply the induction hypothesis for $\mathcal{C}$, $\mathcal{W}$, and σ to the resulting matrix.

$M = \Phi^E : \Phi^M \Downarrow \psi_1^M$ or $M = \Phi^E : \Phi^M \Downarrow \psi_1^E$:

After an application of ψ^M or ψ^E the induction hypothesis can be applied for $\mathcal{C}$, $\mathcal{W}$, and a substitution σ' that results from σ by deleting ψ_1^M (or ψ_1^E) from all images. Note that ψ^E is applicable because of Lemma 24.

$M = \Phi^E : \Phi^M \Downarrow \beta$:

Because of Lemma 23 M has exactly one connected component. Removing β from M yields two connected components. Thus there is an application of the β-rule, which results in two matrices M_1 and M_2 that are complementary for $\mathcal{C}_i = \mathit{FCons}(M_i, \mathcal{C})$, $\mathcal{W}_i = \mathit{FWeak}(M_i, \mathcal{W})$, and σ. To each submatrix M_i we can apply the induction hypothesis for $\mathcal{C}_i$, $\mathcal{W}_i$, and σ. □

To prove the completeness of the matrix characterization, we construct from a Σ_{pos}-proof $\mathcal{P}$ of a matrix M a set $\mathit{ConSet}(\mathcal{P})$ of connections, a weakening map $\mathit{WeakMap}(\mathcal{P})$, and a relation $\sqsubset_{\mathcal{P}} \subseteq \mathit{SpecPos}(M)^2$. We will show that M is complementary for $\mathit{ConSet}(\mathcal{P})$, $\mathit{WeakMap}(\mathcal{P})$ and a substitution $\sigma_{\mathcal{P}}$, which we will construct from $\sqsubset_{\mathcal{P}}$.

$\mathit{ConSet}(\mathcal{P})$ is constructed from applications of *axiom* and τ in $\mathcal{P}$. If *axiom* is applied to $\Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot$ in $\mathcal{P}$ then the connection $\{\mathit{succ}_1(\phi_1^M), \mathit{succ}_1(\phi_2^M)\}$ is in $\mathit{ConSet}(\mathcal{P})$. If τ is applied to a sequent $\Phi^E : \phi_1^M \Uparrow \cdot$ then $\{\mathit{succ}_1(\phi_1^M)\}$ is in $\mathit{ConSet}(\mathcal{P})$.

$\mathit{WeakMap}(\mathcal{P})$ contains those elements of $\mathit{WeakPos}(M)$ that are explicitly weakened by ω or ν and the elements from $\Phi^E(M)$ that are implicitly weakened in *axiom* or τ.

$\sqsubset_{\mathcal{P}}$ represents the order in which special positions are reduced. $p \sqsubset_{\mathcal{P}} p'$ holds if p is reduced before p', i.e. if the reduction occurs closer to the root of the proof tree $\mathcal{P}$. Note that $\sqsubset_{\mathcal{P}}$ is irreflexive, antisymmetric, and transitive, thus an ordering, and that the tree ordering $\ll$ of M, restricted to special positions, is a sub-ordering of $\sqsubset_{\mathcal{P}}$.

LEMMA 26. *Let M be a matrix, $\mathcal{P}$ be a Σ_{pos}-proof for M, and $p_1 \neq p_2 \in (\Psi^M(M) \cup \Psi^E(M))$ be positions in M. If $p_1 \sqsubset_{\mathcal{P}} p$ and $p_2 \sqsubset_{\mathcal{P}} p$ holds for some $p \in \mathit{SpecPos}(M)$, then either $p_1 \sqsubset_{\mathcal{P}} p_2$ or $p_2 \sqsubset_{\mathcal{P}} p_1$.*

We define a mapping $\sigma_{\mathcal{P}} : (\Phi^M \cup \Phi^E) \to (\Psi^M \cup \Psi^E)^*$ as follows:[6] $\sigma_{\mathcal{P}}(\phi)$ is the (wrt. $\sqsubset_{\mathcal{P}}$) ordered sequence $\psi_1 \ldots \psi_n$ of all positions that are reduced before ϕ and after the largest special position $p \ll \phi$.

$\sigma_{\mathcal{P}}(\phi)$ is well-defined, as there is only one way to order the sequence $\psi_1 \ldots \psi_n$ (Lemma 26), and has the following important properties:

sortedness: $\psi_i \sqsubset_{\mathcal{P}} \psi_j$ holds for all $i < j \in \{1 \ldots n\}$.

prior reduction: $\psi_i \sqsubset_{\mathcal{P}} \phi$ holds for all $i \in \{1 \ldots n\}$.

exclusivity: $p \sqsubset_{\mathcal{P}} \psi_i$ holds for all special positions $p \ll \phi$ $(i \in \{1 \ldots n\})$.

maximality: If a position ψ with $\psi \sqsubset_{\mathcal{P}} \phi$ does not occur in $\psi_1 \ldots \psi_n$ then $\psi \ll \phi$ or there is a special position $p \ll \phi$ such that $\psi \sqsubset_{\mathcal{P}} p$.

LEMMA 27. *Let ψ and u be special positions of type $\Psi^M \cup \Psi^E$.*

1. *If $\psi \sqsubset_{\mathcal{P}} u$ then there is a position $\phi \ll u$ with $\psi \sqsubset_{\mathcal{P}} \phi$.*
2. *$\sigma_{\mathcal{P}}(pre_M(u))$ is the unique sequence of all special positions ψ with $\psi \sqsubset_{\mathcal{P}} u$ or $\psi = u$ that is ordered wrt. $\sqsubset_{\mathcal{P}}$.*

Proof.

1. Assume $\psi \sqsubset_{\mathcal{P}} u$. Then there is a branch in $\mathcal{P}$ on which ψ is reduced before u. ψ can only be reduced by one of the rules ψ^M or ψ^E. The rule ψ^M (ψ^E) is only applicable if all other positions in the sequent have a type in Φ^M, Φ^E (in Φ^E).

   If $\psi \ll u$, then there must a position ϕ with $\psi \ll \phi \ll u$ according to the rewrite rules. Otherwise, either u or some predecessor ϕ of u must be contained in the context of the rule application by which ψ is reduced, as u is reduced on the same branch as ψ. Hence, there must be a position $\phi \ll u$ with $\psi \sqsubset_{\mathcal{P}} \phi$.

2. We proceed by induction on $pre_M(u)$, considering two base cases.

   Base case 1: If $pre_M(u) = u$ then there cannot be a ψ such that $\psi \sqsubset_{\mathcal{P}} u$ because of Lemma 27.1. Moreover $\sigma_{\mathcal{P}}(pre_M(u)) = u$.

   Base case 2: If $pre_M(u) = \phi.u$ for some position ϕ then $\sigma_{\mathcal{P}}(\phi.u)$ is ordered because $\sigma_{\mathcal{P}}(\phi)$ is ordered and $\psi' \sqsubset_{\mathcal{P}} \phi \sqsubset_{\mathcal{P}} u$ holds for all ψ' in $\sigma_{\mathcal{P}}(\phi)$. Furthermore, if $\psi \sqsubset_{\mathcal{P}} u$ holds for some ψ, then $\psi \sqsubset_{\mathcal{P}} \phi$ holds according to part 1 and because of maximality ψ occurs in $\sigma_{\mathcal{P}}(\phi)$.

   Step case: If $pre_M(u) = t.\hat{\psi}.\phi.u$ for some positions ϕ and $\hat{\psi}$ and some string t of special positions, then $\sigma_{\mathcal{P}}(t.\hat{\psi}.\phi.u)$ is ordered, because $\sigma_{\mathcal{P}}(t.\hat{\psi})$ and $\sigma_{\mathcal{P}}(\phi)$ are ordered, $\hat{\psi} \sqsubset_{\mathcal{P}} \psi' \sqsubset_{\mathcal{P}} \phi$ holds for all ψ'

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[6] In the following, positions whose names begin with ψ (or ϕ) are assumed to be of type $\Psi^M \cup \Psi^E$ (or $\Phi^M \cup \Phi^E$) unless explicitly stated otherwise.

in $\sigma_{\mathcal{P}}(\phi)$, and $\phi \sqsubset_{\mathcal{P}} u$ holds. Furthermore, if $\psi \sqsubset_{\mathcal{P}} u$ holds for some ψ, then there is some $\phi' \ll u$ with $\psi \sqsubset_{\mathcal{P}} \phi'$. If $\phi' \ll \phi$, then $\psi \sqsubset_{\mathcal{P}} \hat{\psi}$ and ψ occurs in $\sigma_{\mathcal{P}}(t.\hat{\psi})$ by induction hypothesis. Otherwise $\psi \sqsubset_{\mathcal{P}} \phi$ and ψ occurs in $\sigma_{\mathcal{P}}(\phi)$ because of maximality. $\square$

LEMMA 28. *$\sigma_{\mathcal{P}}$ is admissible for $\mathcal{M}$, i.e. if $tu \underline{\angle} \sigma_{\mathcal{P}}(pre_M(v))$ for some $u \in (\Psi^M \cup \Psi^E)$, $t \in (\Psi^M \cup \Psi^E)^*$, and $v \in (\Phi^M \cup \Phi^E \cup \Psi^M \cup \Psi^E)$ then $tu = \sigma_{\mathcal{P}}(pre_M(u))$.*

Proof. Assume that $tu \underline{\angle} \sigma_{\mathcal{P}}(pre_M(v))$ holds. From Lemma 27.2 we obtain that $\sigma_{\mathcal{P}}(pre_M(v))$ is ordered wrt. $\sqsubset_{\mathcal{P}}$ and contains all positions ψ with $\psi \sqsubset_{\mathcal{P}} v$ or $\psi = v$. Hence, the sub-sequence tu also is ordered and contains all positions ψ for which $\psi \sqsubset_{\mathcal{P}} u$ or $\psi = u$ holds, which means that $\sigma_{\mathcal{P}}(pre_M(u)) = tu$ according to Lemma 27.2. $\square$

THEOREM 29 (Completeness).
Let $\mathcal{P}$ be a Σ_{pos}-proof for a matrix M. Then M is complementary for ConSet($\mathcal{P}$), WeakMap($\mathcal{P}$), and $\sigma_{\mathcal{P}}$.

Proof. We prove by induction over the structure of $\mathcal{P}$ that the five requirements for the complementarity of M are satisfied.

Base case: $\mathcal{P}$ consists only of a single rule application of one of the rules *axiom* or τ. In the first case, $M = \Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot$ and $\mathcal{P} = \{ \overline{\ M\ }^{\ axiom}$. Thus $ConSet(\mathcal{P}) = \{\{succ_1(\phi_1^M), succ_1(\phi_2^M)\}\}$, $WeakMap(\mathcal{P}) = \Phi^E$, and $\sqsubset_{\mathcal{P}} = \emptyset$ by construction.

1. $ConSet(\mathcal{P})$ spans $\mathcal{M}$, because $succ_1(\phi_1^M), succ_1(\phi_2^M) \in P$ holds for all $P \in LPaths(M)$ according to Definition 13.
2. $\sigma_{\mathcal{P}}$ unifies $ConSet(\mathcal{P})$, $WeakMap(\mathcal{P})$, since $\sigma_{\mathcal{P}}(pre_M(succ_1(\phi_1^M))) = \sigma_{\mathcal{P}}(pre_M(succ_1(\phi_2^M))) = \varepsilon$ and $\sigma_{\mathcal{P}}(pre_M(\phi^E)) = \varepsilon$ for all $\phi^E \in \Phi^E$.
3. $ConSet(\mathcal{P})$, $WeakMap(\mathcal{P})$ is linear, since $ConSet(\mathcal{P})$ contains only one connection and $WeakMap(\mathcal{P})$ contains no predecessor of a position from that connection.
4. $ConSet(\mathcal{P})$, $WeakMap(\mathcal{P})$ has the relevance property for M, because every leaf position is a successor of of ϕ_1, ϕ_2, or some $\phi^E \in \Phi^E$.
5. $ConSet(\mathcal{P})$, $WeakMap(\mathcal{P})$ has the cardinality property for M since
$$\begin{aligned} & |ConSet(\mathcal{P})| \\ = \ & |\boldsymbol{\beta}(M)| - |\boldsymbol{\beta}(M)| + 1 \\ = \ & |\boldsymbol{\beta}(M)| - \Sigma_{\phi^E \in \Phi^E} |\boldsymbol{\beta}(\phi^E)| + 1 \\ = \ & |\boldsymbol{\beta}(M)| - \Sigma_{\phi^E \in WeakMap(\mathcal{P})} |\boldsymbol{\beta}(\phi^E)| + 1 \end{aligned}$$

The proof is similar if $\mathcal{P}$ consists of an application of the τ rule.

Induction step: We assume that the five properties hold for any subproof $\mathcal{P}'$ of $\mathcal{P}$ and consider the last rule applied in $\mathcal{P}$.

$o \Uparrow$: Assume $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{M'}}{M}\, {\scriptstyle o\Uparrow} \right.,$ where $M = \Phi^E : \Phi^M \Uparrow \Xi, o$ and $M' = \Phi^E : \Phi^M \Uparrow \Xi, \mathit{succ}_1(o)$. Then $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$, $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}')$, $\sqsubset_{\mathcal{P}} = \sqsubset_{\mathcal{P}'}$, and $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'}$.

$\mathit{ConSet}(\mathcal{P})$ spans $\mathcal{M}$ since $\mathit{LPaths}(M) = \mathit{LPaths}(M')$. $\sigma_{\mathcal{P}}$ unifies $\mathit{ConSet}(\mathcal{P}), \mathit{WeakMap}(\mathcal{P})$, since $\mathit{pre}_M(\mathit{succ}_1(p)) = \mathit{pre}_{M'}(\mathit{succ}_1(p))$ holds for all $p \in \mathit{LeafPos}(M) \cup \Phi^E(M)$. Linearity and relevance follow directly from the induction hypothesis. Cardinality holds because $|\boldsymbol{\beta}(M)| = |\boldsymbol{\beta}(M')|$, $|\mathit{ConSet}(\mathcal{P})| = |\mathit{ConSet}(\mathcal{P}')|$, and $\sum_{\phi^E \in \mathit{WeakMap}(\mathcal{P})} |\boldsymbol{\beta}(\phi^E)| = \sum_{\phi^E \in \mathit{WeakMap}(\mathcal{P}')} |\boldsymbol{\beta}(\phi^E)|$.

$o \Downarrow$, α, π, ϕ^M : These cases can be shown similarly to the case for $o \Uparrow$.

focus_1 : Assume $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{M'}}{M}\, {\scriptstyle \mathit{focus}_1} \right.,$ where $M = \Phi^E, \phi^E_1 : \Phi^M \Uparrow \cdot$, and $M' = \Phi^E : \Phi^M \Downarrow \mathit{succ}_1(\phi^E_1)$. Then $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$, $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}')$, and $\sqsubset_{\mathcal{P}} = \sqsubset_{\mathcal{P}'} \cup \{(\phi^E_1, p) \mid p \in \mathit{SpecPos}(M')\}$.

The spanning, linearity, relevance, and cardinality properties follow from the from the induction hypothesis as in the case for $o \Uparrow$. $\sigma_{\mathcal{P}}$ unifies $\mathit{ConSet}(\mathcal{P}), \mathit{WeakMap}(\mathcal{P})$, because $\sigma_{\mathcal{P}}(\mathit{pre}_M(p)) = \sigma_{\mathcal{P}'}(\mathit{pre}_{M'}(p))$ holds for all $p \in \mathit{LeafPos}(M) \cup \Phi^E(M)$: if $p \in \mathit{Pos}(\phi^E_1)$ then $\mathit{pre}_M(p) = \phi^E_1.\mathit{pre}_{M'}(p)$ and $\sigma_{\mathcal{P}}(\phi^E_1.\mathit{pre}_{M'}(p)) = \sigma_{\mathcal{P}'}(\mathit{pre}_{M'}(p))$ by prior reduction. Otherwise $\mathit{pre}_M(p) = \mathit{pre}_{M'}(p)$.

focus_2 : This case can be shown similarly to the case for focus_1.

ψ^M : Assume $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{M'}}{M}\, {\scriptstyle \psi^M} \right.,$ where $M = \Phi^E : \Phi^M \Downarrow \psi^M_1$, and $M' = \Phi^E : \Phi^M \Uparrow \mathit{succ}_1(\psi^M_1)$. Then $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$, $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}')$, and $\sqsubset_{\mathcal{P}} = \sqsubset_{\mathcal{P}'} \cup \{(\psi^M_1, p) \mid p \in \mathit{SpecPos}(M')\}$.

The spanning, linearity, relevance, and cardinality properties follow from the from the induction hypothesis as in the case for $o \Uparrow$. $\sigma_{\mathcal{P}}$ unifies $\mathit{ConSet}(\mathcal{P}), \mathit{WeakMap}(\mathcal{P})$, because $\sigma_{\mathcal{P}}(\mathit{pre}_M(p)) = \psi^M_1.\sigma_{\mathcal{P}'}(\mathit{pre}_{M'}(p))$ holds for all $p \in \mathit{LeafPos}(M) \cup \Phi^E(M)$: if $p \in \mathit{Pos}(\psi^M_1)$ then $\mathit{pre}_M(p) = \psi^M_1.\mathit{pre}_{M'}(p)$ and $\sigma_{\mathcal{P}}(\psi^M_1.\mathit{pre}_{M'}(p)) = \psi^M_1.\sigma_{\mathcal{P}'}(\mathit{pre}_{M'}(p))$. Otherwise $\mathit{pre}_M(p) = \mathit{pre}_{M'}(p)$ and $\sigma_{\mathcal{P}}(\mathit{pre}_M(p)) = \psi^M_1.\sigma_{\mathcal{P}'}(\mathit{pre}_{M'}(p))$ by Lemma 27.2.

ψ^E : This case can be shown similarly to the case for ψ^M.

ω : Assume $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{M'}}{M}\,{}_{\omega}\right.$, where $M = \Phi^E : \Phi^M \Uparrow \Xi, \omega$, and $M' = \Phi^E : \Phi^M \Uparrow \Xi$. Then $\mathit{LeafPos}(M) = \mathit{LeafPos}(M') \cup \{\omega\}$, $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$, $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}') \cup \{\omega\}$, $\sqsubset_{\mathcal{P}} = \sqsubset_{\mathcal{P}'}$, and $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'}$.

$\mathit{ConSet}(\mathcal{P})$ spans $\mathcal{M}$ because for all paths $P \in \mathit{LPaths}(M)$ there is a path $P' \in \mathit{LPaths}(M')$ with $P = P' \cup \{\omega\}$. $\sigma_{\mathcal{P}}$ unifies $\mathit{ConSet}(\mathcal{P})$, $\mathit{WeakMap}(\mathcal{P})$, since $\mathit{pre}_{M}(\mathit{succ}_1(p)) = \mathit{pre}_{M'}(\mathit{succ}_1(p))$ holds for all $p \in \mathit{LeafPos}(M) \cup \Phi^E(M)$ with $p \neq \omega$ and $\mathit{pre}_{M}(\mathit{succ}_1(p)) = \varepsilon$ holds for $p = \omega$. The linearity, relevance, and cardinality properties follow from the induction hypothesis as in the case for $o \Uparrow$.

ν : This case is similar to the case for the ω rule if $\mu(\nu) = 0$ and like the case for the $o \Uparrow$ rule if $\mu(\nu) > 0$.

β : Assume $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{M'} \qquad \dfrac{\mathcal{P}''}{M''}}{M}\,{}_{\beta}\right.$, where $M = \Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta$, $M' = \Phi_1^E : \Phi_1^M \Downarrow \mathit{succ}_1(\beta)$, and $M'' = \Phi_2^E : \Phi_2^M \Downarrow \mathit{succ}_2(\beta)$.

Then $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}') \cup \mathit{ConSet}(\mathcal{P}'')$ and $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}') \cup \mathit{WeakMap}(\mathcal{P}'')$.

Let $P \in \mathit{LPaths}(M)$. Then either $(\mathit{Pos}(\beta) \cap P) \subseteq \mathit{Pos}(\mathit{succ}_1(\beta))$ or $(\mathit{Pos}(\beta) \cap P) \subseteq \mathit{Pos}(\mathit{succ}_2(\beta))$. In the first case $P' \subseteq P$ for some path $P' \in \mathit{LPaths}(M')$ and thus there is a connection $C' \in \mathit{ConSet}(\mathcal{P}')$ with $C' \subseteq P'$. In the other case $P'' \subseteq P$ for some $P'' \in \mathit{LPaths}(M'')$ and there is a connection $C'' \in \mathit{ConSet}(\mathcal{P}'')$ with $C'' \subseteq P''$. Thus $\mathit{ConSet}(\mathcal{P})$ spans $\mathcal{M}$.

$\mathit{ConSet}(\mathcal{P})$, $\mathit{WeakMap}(\mathcal{P})$ is linear, since the weakening maps and connection sets of $\mathcal{P}'$ and $\mathcal{P}''$ are disjoint. Unifiability and relevance follow from the induction hypothesis as in the case for $o \Uparrow$. The cardinality property follows from the following equations:

$$\begin{aligned}
|\boldsymbol{\beta}(M)| &= |\boldsymbol{\beta}(M')| + |\boldsymbol{\beta}(M'')| + 1 \\
|\mathit{ConSet}(\mathcal{P})| &= |\mathit{ConSet}(\mathcal{P}') \cup \mathit{ConSet}(\mathcal{P}'')| \\
\textstyle\sum_{\phi^E \in \mathit{WeakMap}(\mathcal{P})} |\boldsymbol{\beta}(\phi^E)| &= \textstyle\sum_{\phi^E \in \mathit{WeakMap}(\mathcal{P}')} |\boldsymbol{\beta}(\phi^E)| \\
&\quad + \textstyle\sum_{\phi^E \in \mathit{WeakMap}(\mathcal{P}'')} |\boldsymbol{\beta}(\phi^E)| \qquad \square
\end{aligned}$$

Theorems 25 and 29 show that the matrix of a signed formula φ is complementary for some multiplicity μ if and only if φ is derivable in Σ_{pos}. According to Theorem 12 this is the case if and only if φ is derivable in Σ_3', which in turn is equivalent to φ being derivable in Σ_1 according to Theorem 4. As derivability in Σ_1 defines validity in $\mathcal{MELL}$, complementarity is in fact a valid characterization of logical validity in $\mathcal{MELL}$, which proves Theorem 21.

7. Conclusion

We have presented a matrix characterization of logical validity for the multiplicative exponential fragment of linear logic ($\mathcal{MELL}$). It extends our characterization for $\mathcal{MLL}$ (Kreitz et al., 1997) by the exponentials ? and ! and the multiplicative constants **1** and $\perp$. Our extension, as pointed out in (Fronhöfer, 1996), is by no means trivial and goes beyond the existing matrix characterizations for fragments of linear logic.

In the process a methodology has been outlined for developing matrix characterizations from sequent calculi and for proving them correct and complete. It introduces a series of intermediate calculi, which stepwisely remove redundancies from sequent proofs while capturing their essential parts, and arrives at a matrix characterization as the most compact representation for proof search.

If applied to modal or intuitionistic logics, this methodology would essentially lead to Wallen‘s matrix characterization (Wallen, 1990). In order to capture the resource sensitivity of linear logic, however, several refinements have been introduced. The notion of multiplicities is based on positions instead of basic positions. Different types of special positions are used. The novel concept of weakening maps makes us able to deal with the aspects of resource management. In linear logic, weakening can only be applied on certain formulas. A matrix proof must ensure that it is possible to weaken all positions which take not part in an axiom in the corresponding sequent proofs. This is ensured by weakening maps together with a modified unifiability requirement.

Fronhöfer has developed matrix characterizations for various variations of the multiplicative fragment of linear logic (Fronhöfer, 1996). Compared to his work for linear logic our characterization captures additionally the multiplicative constants and the controlled application of weakening and contraction. In fact, we are confident that our methodology will extend to further fragments of linear logic as well as to other resource sensitive logics, such as affine or relevance logics.

Galmiche and colleagues have investigated procedures for automated proof net construction various fragments of linear logic and analyzed the relationship between proof nets and matrix methods. The investigations include $\mathcal{MLL}$ (Galmiche and Perrier, 1994; Galmiche, 2000); its extension $\mathcal{MNL}$ which includes non-commutative conjunction and disjunction (Galmiche and Notin, 2000; Galmiche and Notin, 2001; Galmiche and Notin, 2002); and a separate approach to additive linear logic $\mathcal{ALL}$ (Galmiche and Marion, 1995; Galmiche and Martin, 1997; Galmiche, 2000). However, it is not clear yet how to integrate the approaches to $\mathcal{MLL}$ and $\mathcal{ALL}$, and exponentials are captured by neither of the approaches.

Because of the many benefits of matrix characterizations for automated theorem proving, it would be desirable to extend our matrix characterization of logical validity to full linear logic. Although this may require the development of more novel concepts, we believe that such an extension will be possible. The main challenge, as discussed in Section 5.3, is the adaption of prefixes to the additives such that they can be used in combination with the prefixes for multiplicatives..

Another direction is to extend our characterization to quantifiers. Although much is known about quantifiers in other logics, this again is a non-trivial problem, as we have to investigate the interactions between term- and prefix-substitutions.

A third area could be to reduce the number of special positions that need to be inserted. Inserting special positions between two positions of type π does not seem to be essential. A similar observation can be made for two adjacent positions of type ν. Also for literal positions some special positions could be removed following Andreoli's technique to handle positive and negative literals differently in the axiom rule (Andreoli, 1993). However, these optimizations will complicate the already complex exposition of matrix characterizations even further and may not result in significant improvements in practice.

The matrix characterization presented here is a condensed representation of the search space. In general, matrix characterizations are known as a foundation for efficient proof search procedures for classical, modal and intuitionistic logics (Kreitz and Otten, 1999) and $\mathcal{MLL}$ (Kreitz et al., 1997). We expect that these proof procedures can now be extended to $\mathcal{MELL}$ and a wide spectrum of other logics, as soon as our methodology has led us to a matrix characterization for them. Experimental results with the tableau prover linTAP (Mantel and Otten, 1999) for a sub-fragment of $\mathcal{MELL}$, whose underlying theory is based on the results presented in this article, have been very promising.

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IDENTITY

$$\frac{}{\Theta : \langle A, + \rangle, \langle A, - \rangle}\ \mathit{axiom}$$

NEGATION

$$\frac{\Theta : \Upsilon, \mathit{succ}_1(o)}{\Theta : \Upsilon, o}\ o$$

MULTIPLICATIVE FRAGMENT

$$\frac{}{\Theta : \tau}\ \tau \qquad \frac{\Theta : \Upsilon}{\Theta : \Upsilon, \omega}\ \omega$$

$$\frac{\Theta : \Upsilon, \mathit{succ}_1(\alpha), \mathit{succ}_2(\alpha)}{\Theta : \Upsilon, \alpha}\ \alpha \qquad \frac{\Theta_1 : \Upsilon_1, \mathit{succ}_1(\beta) \quad \Theta_2 : \Upsilon_2, \mathit{succ}_2(\beta)}{\Theta_1, \Theta_2 : \Upsilon_1, \Upsilon_2, \beta}\ \beta$$

EXPONENTIAL FRAGMENT

$$\frac{\Theta, \mathit{succ}_1(\nu)^{\mu(\nu)} : \Upsilon}{\Theta : \Upsilon, \nu}\ \nu \qquad \frac{\Theta : \mathit{succ}_1(\pi)}{\Theta : \pi}\ \pi$$

FOCUSING

$$\frac{\Theta : \Upsilon, \varphi}{\Theta, \varphi : \Upsilon}\ \mathit{focus}$$

Figure 8. The dyadic sequent calculus Σ_2' for $\mathcal{MELL}$

Appendix

A. Soundness and Completeness of Σ_3'

In this appendix, we prove the soundness and completeness of Σ_3', our variant of Andreoli's triadic sequent calculus Σ_3 (Andreoli, 1993). Our proofs are essentially along the same lines as the corresponding proofs for Σ_3, but differ in some technical details. In the proofs, we proceed as follows: Firstly, we introduce a variant Σ_2' of Andreoli's dyadic calculus Σ_2, which serves as intermediate step for proving Theorem 4. It differs from Σ_2 in the way structural rules are handled – similar to the difference between Σ_3' and Σ_3. Secondly, we prove Σ_2' sound and complete based on the soundness and completeness of Σ_1 and Σ_1'. Thirdly, we prove that Σ_3' is sound and complete based on the results for Σ_2'.

As depicted in Figure 8, our dyadic calculus Σ_2' operates on *dyadic sequents*, i.e. on multi-sets of signed formulas where each formula is associated with one of two zones, an *unbounded zone* or a *bounded zone*. Unlike in triadic sequents, there is no focused zone and also no mode. Formally, a dyadic sequent has the form $\Theta : \Upsilon$ where Θ, the *unbounded zone*, and Υ, the *bounded zone*, are multi-sets of signed formulas. The fact that a given formula is in a particular zone encodes information about this formula. The rules of Σ_2' ensure that all formulas in the unbounded zone have originated from generic formulas. This requirement is not imposed on formulas in the bounded zone. To reduce a formula in the dyadic calculus, it must be in the bounded zone. This is the purpose of the *focus*-rule in Σ_2'.

Derivations of a dyadic sequent are defined with respect to a *fixed* multiplicity function μ. A signed formula φ is *derivable in* Σ_2' if $\cdot : \varphi$ is derivable for some multiplicity, where $\cdot$ denotes the empty multi-set.

We show that every Σ_2'-proof can be transformed into a Σ_1-proof.

LEMMA 30. *If a dyadic sequent* $S_2 = \Theta{:}\Upsilon$ *has a* Σ_2'*-proof with multiplicity* μ_2 *then the unary sequent* $S_1 = \, ?\Theta^+, !\Theta^-, \Upsilon$ *is derivable in* Σ_1 *where* Θ^+ *and* Θ^- *contain all positive and negative formulas from* Θ.[7]

Proof. Let $\mathcal{P}_2$ be the Σ_2'-proof for S_2. We construct a Σ_1-proof $\mathcal{P}_1$ for S_1 by induction over the structure of $\mathcal{P}_2$. If $\mathcal{P}_2$ consists only of an application of *axiom* or τ then we construct a Σ_1-proof for S_1 by first weakening all formulas in $?\Theta^+$ and $!\Theta^-$ (rule w) and then applying *axiom* or τ. Otherwise, we construct $\mathcal{P}_1$ by transforming the Σ_2'-rule that is applied at the root of $\mathcal{P}_2$ and then applying the induction hypothesis to complete the Σ_1-proof. If the rule at the root of $\mathcal{P}_2$ is o, ω, α, β, or π, we apply the corresponding Σ_1-rule to S_1. If the ν-rule has been applied to some formula ν then we apply w to ν if $\mu(\nu){=}0$, apply no rule if $\mu(\nu){=}1$, and apply the contraction rule c for $(\mu_2(\nu){-}1)$-times (if $\mu(\nu){>}1$). If the rule is *focus* then we apply ν to S_1. $\square$

We show that every Σ_1'-proof can be transformed into a Σ_2'-proof.

LEMMA 31. *If the unary sequent* $S_1 = \Upsilon$ *has a* Σ_1'*-proof with multiplicity* μ_1 *then there is a multiplicity* μ_2 *for which the dyadic sequent* $S_2 = \mathit{succ}_1(\Upsilon_\nu)^{\mu_2} : \Upsilon_{\overline{\nu}}$ *is derivable in* Σ_2'*, where for each* ν*-formula* ν *in* Υ*,* $\mathit{succ}_1(\Upsilon_\nu)^{\mu_2}$ *contains* $\mu_2(\nu)$ *copies of the main sub-formula of* ν *and* $\Upsilon_{\overline{\nu}}$ *contains all positions from* Υ *that are* not *of type* ν*.*

Proof. Let $\mathcal{P}_1$ be the Σ_1'-proof of S_1. We construct a Σ_2'-proof $\mathcal{P}_2$ for S_2 by induction over the structure of $\mathcal{P}_1$. For a sub-formula ν_2 of S_2 we define $\mu_2(\nu_2) = \mu_1(\nu_1)$. If $\mathcal{P}_1$ consists only of an application of *axiom* or τ then $\mathcal{P}_2$ consists of an application of *axiom* or τ, respectively.

Otherwise, we construct $\mathcal{P}_2$ by transforming the Σ_1'-rule at the root of $\mathcal{P}_1$ and then applying the induction hypothesis to complete the Σ_2'-proof. We make a case distinction depending on the number n of active formulas of the rule at the root of $\mathcal{P}_1$ that are of type ν, showing the case $n{=}0$ in detail. The proofs of the other cases are similar with the only difference that the ν-rule must be applied in $\mathcal{P}_2$ to all active formulas of type ν before the induction hypothesis can be applied.

---

[7] The notation $f(\Theta)$ abbreviates $\{f(e) \mid e{\in}\Theta\}$, if f is defined on elements of the multi-set Θ. For a unary connective c, $c\Theta$ abbreviates $\{\langle cF, k\rangle \mid \langle F, k\rangle{\in}\Theta\}$.

Assume that *no* ν-formulas are active in the rule at the root of $\mathcal{P}_1$. If the last Σ_1'-rule is o, ω, α, β, or π, we apply the respective Σ_2'-rule to S_2 and obtain a proof for the resulting premise by induction hypothesis. If the rule is ν, we apply the *focus* rule to the corresponding major subformula $succ_1(\nu)$ of ν in the unbounded zone of S_2. If c_{μ_1} is applied to a formula ν at the root of $\mathcal{P}_1$, we apply no rule at all (according to our construction, formulas of type ν are immediately contracted and moved into the unbounded zone when they occur in a sequent). $\square$

THEOREM 32. *A signed formula φ is derivable in Σ_2' for some multiplicity if and only if it is derivable in Σ_1.*

Proof. If $\cdot : \varphi$ is derivable in Σ_2' for the multiplicity μ_2 then φ is derivable in Σ_1 because of Lemma 30. If φ is derivable in Σ_1 then φ is also derivable in Σ_1' for some multiplicity μ_1 (cf. Section 2.3). Let $\mathcal{P}_1$ be a Σ_1'-proof of φ. If φ is not of type ν then $\cdot : \varphi$ is derivable in Σ_2' for some multiplicity μ_2 according to Lemma 31. Otherwise (φ is of type ν) $succ_1(\varphi)^{\mu(\varphi)} : \cdot$ is derivable in Σ_2' for some μ_2. Hence, we first have to apply the rule ν in order to reduce $\cdot : \varphi$ to $succ_1(\varphi)^{\mu(\varphi)} : \cdot$ before we can apply Lemma 31. $\square$

We show that every Σ_3'-proof can be transformed into a Σ_2'-proof.

LEMMA 33. *If a triadic sequent $S_3 = \ \Theta : \Upsilon \Uparrow \Xi$ (or $\Theta : \Upsilon \Downarrow \varphi$) has a Σ_3'-proof with multiplicity μ_3 then there is a multiplicity μ_2 such that the dyadic sequent $S_2 = \ \Theta : \Upsilon, \Xi$ (or $\Theta : \Upsilon, \varphi$) is derivable in Σ_2'.*[8]

Proof. Let $\mathcal{P}_3$ be the Σ_3'-proof of S_3. We construct a Σ_2'-proof $\mathcal{P}_2$ for S_2 by induction over the structure of $\mathcal{P}_3$. If $\mathcal{P}_3$ consists of an application of *axiom* or τ then $\mathcal{P}_2$ consists of the same rule in Σ_2'. Otherwise, we construct $\mathcal{P}_2$ by analyzing the Σ_3'-rule at the root of $\mathcal{P}_3$. If this rule is $o \Downarrow$, $o \Uparrow$, ω, α, β, ν, or π we apply the corresponding Σ_2'-rule and obtain a Σ_2'-proof for S_2 by induction hypothesis. For *focus*$_1$ we apply *focus* and, again, obtain a Σ_2'-proof for S_2 by induction hypothesis. For *focus*$_2$, *defocus*, or *switch* we immediately have a Σ_2'-proof by induction hypothesis and need not apply any further rule. $\square$

The following inversion lemma gives us the necessary freedom to treat a sequence of signed formulas in the focused zone of a triadic sequent as if it were a multi-set. It is helpful for proving completeness of Σ_3'.

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[8] We deliberately take some freedom in the notation here by regarding the sequence Ξ as a multi-set. This is justified by Lemma 34(3).

LEMMA 34 (Inversion). *Let* $\Theta, \Theta_1, \Theta_2$, $\Upsilon, \Upsilon_1, \Upsilon_2$ *be multi-sets of signed formulas and* Ξ, Ξ_1, Ξ_2 *be sequences of signed formulas.*

(1) *If* $\Theta_1 : \Upsilon_1 \Uparrow succ_1(\beta), \Xi_1$ *and* $\Theta_2 : \Upsilon_2 \Uparrow succ_2(\beta), \Xi_2$ *can be derived in* Σ_3' *then so can* $\Theta_1, \Theta_2 : \Upsilon_1, \Upsilon_2, \beta \Uparrow \Xi_1, \Xi_2$.

(2) *If* $\Theta : \Upsilon \Uparrow \varphi, \Xi$ *can be derived in* Σ_3' *then so can* $\Theta, \varphi : \Upsilon \Uparrow \Xi$.

(3) *If* $\Theta : \Upsilon \Uparrow \Xi$ *can be derived in* Σ_3' *and* $\Xi \equiv \Xi'$, *i.e. if* Ξ *and* Ξ' *differ only in the order of elements, then* $\Theta : \Upsilon \Uparrow \Xi'$ *can be derived in* Σ_3'.

Proof. Similar to the proofs of inversion lemmas in (Andreoli, 1993).

We show that every Σ_3'-proof can be transformed into a Σ_2'-proof.

LEMMA 35. *If a dyadic sequent* $S_2 = \ \Theta : \Upsilon$ *has a* Σ_2'*-proof for some multiplicity* μ_2 *then there is a multiplicity* μ_3 *such that the triadic sequent* $S_3 = \ \Theta : \cdot \Uparrow \Upsilon$ *is derivable in* Σ_3'.

Proof. Let $\mathcal{P}_2$ be the Σ_2'-proof of S_2. We construct a Σ_3'-proof $\mathcal{P}_3$ for S_3 by induction over the structure of $\mathcal{P}_2$. Because of Lemma 34(3) we can assume the main formula to be the last formula in the focused zone. For any sub-formula φ_2 of S_2 for which μ_2 is defined we set $\mu_3(\varphi_3) = \mu_2(\varphi_2)$, where φ_3 is the corresponding sub-formula in S_3. If $\mathcal{P}_2$ consists of *axiom* or τ, we construct $\mathcal{P}_3$ by applying *defocus* to all formulas in the focused zone in S_3 and then applying *axiom* or τ.

Otherwise, we analyze the Σ_2'-rule at the root of $\mathcal{P}_2$. If it is o, ω, α, or ν, we apply the corresponding Σ_3'-rule to S_3 (use $o \Uparrow$ for o) and obtain a Σ_3'-proof for S_3 by induction hypothesis. If it is a β-rule with premises $\Theta_1 : \Upsilon_1, succ_1(\beta)$ and $\Theta_2 : \Upsilon_2, succ_2(\beta)$, we have Σ_3'-proofs for $\Theta_1 : \cdot \Uparrow \Upsilon_1, succ_1(\beta)$ and $\Theta_2 : \cdot \Uparrow \Upsilon_2, succ_2(\beta)$ by induction hypothesis. Due to Lemma 34(1) the sequent $\Theta_1, \Theta_2 : \beta \Uparrow \Upsilon_1, \Upsilon_2$, the premise of applying *defocus* to $S_3 = \ \Theta_1, \Theta_2 : \cdot \Uparrow \Upsilon_1, \Upsilon_2, \beta$, is derivable in Σ_3' and so is S_3. If the rule at the root of $\mathcal{P}_2$ is π, we apply *defocus*, *focus*$_2$, and π to S_3 and obtain a Σ_3'-proof for S_3 by induction hypothesis. If it is *focus* with premise $\Theta : \Upsilon, \varphi$ then $\Theta : \cdot \Uparrow \Upsilon, \varphi$ is derivable by induction hypothesis. By Lemma 34(2) $S_3 = \ \Theta, \varphi : \cdot \Uparrow \Upsilon$ is derivable in Σ_3'. □

Based on Theorem 32 and Lemmas 33 and 35, we can now prove:

THEOREM 4 (Soundness and Completeness of $\boldsymbol{\Sigma_3'}$). *A signed formula* φ *is derivable in* Σ_3' *if and only if it is derivable in* Σ_1.

Proof. If $\cdot : \cdot \Uparrow \varphi$ is derivable in Σ_3' for μ_3 then we obtain from Lemma 33 that $\cdot : \varphi$ is derivable in Σ_2' for some multiplicity μ_2. From Theorem 32, we obtain that φ is derivable in Σ_1 (soundness).

If φ is derivable in Σ_1 then we obtain from Theorem 32 that $\cdot : \varphi$ is also derivable in Σ_2' for some multiplicity μ_2. From Lemma 35, we obtain that $\cdot : \cdot \Uparrow \varphi$ is derivable in Σ_3' for some multiplicity μ_3. $\square$

B. Soundness and Completeness of Σ_{pos}

We prove the soundness and completeness of Σ_{pos}, the sequent calculus for matrices defined in Section 4, by showing that a signed formula φ is derivable in Σ_{pos} if and only if it is derivable in Σ_3' (Theorem 12).

Proof. To prove the *soundness* of Σ_{pos}, we show that if a matrix for $S = \Phi^E : \Phi^M \Uparrow \Xi$ (or $\Phi^E : \Phi^M \Downarrow \varphi$) is derivable in Σ_{pos} then there is a multiplicity μ_3 such that the corresponding triadic sequent $S_3 = \mathit{sform}(\Phi^E) : \mathit{sform}(\Phi^M) \Uparrow \mathit{sform}(\Xi)$ (or $\mathit{sform}(\Phi^E) : \mathit{sform}(\Phi^M) \Downarrow \mathit{sform}(\varphi)$) is derivable in Σ_3'.

We proceed by induction on the structure of the Σ_{pos}-proof $\mathcal{P}_m$. An application of one of the rules *axiom*, $o \Downarrow$, $o \Uparrow$, τ, ω, α, β, ν, ψ^E, focus_1, focus_2, ϕ^M, or ψ^M in $\mathcal{P}_m$ results in an application of the rule *axiom*, $o \Downarrow$, $o \Uparrow$, τ, ω, α, β, ν, π, focus_1, focus_2, *defocus*, or *switch*, respectively in the proof $\mathcal{P}_3$ of S_3. If π is the last rule applied in $\mathcal{P}_m$ then no corresponding rule application is required in $\mathcal{P}_3$. This follows directly from the definition of *sform* on inserted positions.

To prove the *completeness* of Σ_{pos}, we show that if a triadic sequent $S_3 = \Theta : \Upsilon \Uparrow \Xi$ (or $\Theta : \Upsilon \Downarrow \varphi$) is derivable in Σ_3' then there is a multiplicity such that the corresponding matrix is derivable in Σ_{pos}.

We denote the position tree corresponding to a signed formula φ by $\widetilde{\varphi}$. Similarly, $\widetilde{\Xi}$, $\widetilde{\Theta}$, $\widetilde{\Upsilon}$, and $\widetilde{S_3}$ denote position tree versions of Ξ, Θ, Υ, and S_3. According to the definition of position sequents either $\widetilde{S_3} = \Phi^E : \Phi^M \Uparrow \widetilde{\Xi}$ or $\widetilde{S_3} = \Phi^E : \Phi^M \Downarrow \widetilde{\varphi}$ hold, where $\widetilde{\Xi}$ contains only positions with type in $\{\alpha, \nu, o, \omega, \phi^M\}$ and $\widetilde{\varphi}$ has a type in $\{\beta, \pi, o, \psi^M, \psi^E\}$. We construct a multiplicity $\widetilde{\mu_3}$ and a proof $\widetilde{\mathcal{P}_3}$ for $\widetilde{S_3}$ by induction over the structure of $\mathcal{P}_3$.

Base case: $\mathcal{P}_3$ consists only of a single rule application of one of the rules *axiom* or τ. In the first case $\mathcal{P}_3 = \left\{ \overline{\Theta : \langle A,+\rangle, \langle A,-\rangle \Uparrow \cdot}^{\;\mathit{axiom}} \right.$ and $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\langle A,+\rangle}, \widetilde{\langle A,-\rangle} \Uparrow \cdot$. ϕ_1^M and ϕ_2^M occur at the roots of $\widetilde{\langle A,+\rangle}$ and $\widetilde{\langle A,-\rangle}$ according to the rewrite rule R_{a}. No other special position is inserted into the two trees. $\mathit{Ptype}(\phi_1^M) = \mathit{lit} = \mathit{Ptype}(\phi_2^M)$, $\mathit{lab}(\phi_1^M) = \mathit{lab}(\phi_2^M)$, and $\mathit{pol}(\phi_1^M) \neq \mathit{pol}(\phi_2^M)$. Thus the *axiom* rule is applicable on $\widetilde{S_3}$ and $\widetilde{\mathcal{P}_3} = \left\{ \overline{\widetilde{\Theta} : \phi_1^M, \phi_2^M \Uparrow \cdot}^{\;\mathit{axiom}} \right.$

The proof is similar if $\mathcal{P}_3$ consists of an application of the τ rule.

Induction step: We assume that the theorem holds for any subproof ${\mathcal{P}_3}'$ of $\mathcal{P}_3$ and consider the last rule applied in $\mathcal{P}_3$.

$o \Downarrow$ **:** If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta{:}\Upsilon{\Downarrow}\mathit{succ}_1(o)}}{\Theta{:}\Upsilon{\Downarrow}o}\; o\Downarrow \right.$ then $\widetilde{S_3} = \Phi^E : \Phi^M \Downarrow \widetilde{o}$.

Let $S_3' = \Theta : \Upsilon \Downarrow \mathit{succ}_1(o)$. We show $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \mathit{succ}_1(\widetilde{o})$ and use a case distinction depending on the type r of $\mathit{succ}_1(o)$.

$r \in \{\alpha, \nu, \omega\}$ **:** In this case $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ and, according to R^{φ}_{r}, $\mathit{Ptype}(\mathit{succ}_1(\psi^M)) = r$. The rewrite rule R^{φ}_{r} ensures that the position ψ^M occurs between o and $\mathit{succ}_1(o)$ in $\widetilde{S_3}$.

$r \in \{\beta, \pi, o\}$ **:** In this case $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{r}$ and no position occurs between o and $\mathit{succ}_1(o)$ in $\widetilde{S_3}$.

$r \in \{\tau, a\}$ **:** In this case $\widetilde{S_3}' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ with $\mathit{succ}_1(\psi^M) = \phi^M$ and $\mathit{Ptype}(\mathit{succ}_1(\phi^M)) = r$ according to R^{φ}_{r}. This rewrite rule ensures that the positions ψ^M and ϕ^M occur between o and $\mathit{succ}_1(o)$ in $\widetilde{S_3}$.

By induction hypothesis $\widetilde{S_3'}$ has a proof $\widetilde{\mathcal{P}_3'}$. We can construct a proof for $\widetilde{S_3}$ by an application of $o \Downarrow$: $\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}{:}\widetilde{\Upsilon}{\Downarrow}\mathit{succ}_1(\widetilde{o})}}{\widetilde{\Theta}{:}\widetilde{\Upsilon}{\Downarrow}\widetilde{o}}\; o\Downarrow \right.$

$o \Uparrow$ **:** This case can be shown similarly to the case for $o \Downarrow$.

ω **:** This case is trivial since no rewrite rule inserts special positions at the root of a tree consisting just of an ω-type formula which is in the focused zone in mode $\Uparrow$.

α **:** If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta{:}\Upsilon{\Uparrow}\Xi,\mathit{succ}_1(\alpha),\mathit{succ}_2(\alpha)}}{\Theta{:}\Upsilon{\Uparrow}\Xi,\alpha}\; \alpha \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}, \widetilde{\alpha}$.

Let $S_3' = \Theta : \Upsilon \Uparrow \Xi, \mathit{succ}_1(\alpha), \mathit{succ}_2(\alpha)$. According to the rewrite rules $\mathrm{R}^{\Xi}_{?}$ and $\mathrm{R}^{\alpha}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}, \mathit{succ}_1(\widetilde{\alpha}), \mathit{succ}_2(\widetilde{\alpha})$. Thus, after an application of α on $\widetilde{S_3}$ the induction hypothesis can be applied: $\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}{:}\widetilde{\Upsilon}{\Uparrow}\widetilde{\Xi},\mathit{succ}_1(\widetilde{\alpha}),\mathit{succ}_2(\widetilde{\alpha})}}{\widetilde{\Theta}{:}\widetilde{\Upsilon}{\Uparrow}\widetilde{\Xi},\widetilde{\alpha}}\; \alpha \right.$

β, ν, π, focus_1, focus_2, *defocus*, *switch* **:** These cases can be shown in using a similar argument as above. □